Large deviations for stable like random walks on $\mathbb{Z}^d$ with applications to random walks on wreath products

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Abstract

We derive Donsker-Vardhan type results for functionals of the occupation times when the underlying random walk on $\mathbb{Z}^d$ is in the domain of attraction of an operator-stable law on $\mathbb{R}^d$. Applications to random walks on wreath products (also known as lamplighter groups) are given.

1 Introduction

This work addresses two closely related questions of independent interests. From the point of view of random walks on the lattices $\mathbb{Z}^d$, we extend the well-known large deviation theorem of Donsker and Varadhan regarding the Laplace transform of the number of visited points. The theorem of Donsker and Varadhan, [7], treats random walks driven by measure in the domain of normal attraction of any stable rotationally symmetric law of index $\alpha \in (0, 2)$ (as well as the Gaussian case). We generalize this result to random walks driven by a measure in the domain of attraction of an *operator-stable law*. For instance, this includes laws that are "stable" with respect to inhomogeneous dilations of the type $\delta_t(x_1, \ldots, x_d) = (t^{\alpha_1} x_1, \ldots, t^{\alpha_d} x_d)$.

In addition, if $l(n, x)$ denotes the number of visits at x up to time n, we are interested in obtaining this type of large deviation result for the Laplace transform of more general functionals of the occupation time vector $(l(n, x))_{x \in \mathbb{Z}^d}$ than the number of visited sites, $\#\{x : l(n, x) \neq 0\}$. For instance, we are interested in the asymptotic behavior of

$$-\log \mathbf{E}\left(e^{-\lambda \sum_{x \in \mathbb{Z}^d} \log l(n,x)}\right)$$

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and, more generally,

$$-\log \mathbf{E}\left(e^{-\lambda \sum_{x\in\mathbb{Z}^d} F(l(n,x))}\right)$$

when F belongs to some appropriate class of functions. To obtain results in this direction, we adapt to the operator-stable case the technique introduced in [2]. See also [9].

In developing these results, we are motivated by applications to the study of random walks on a class of groups called wreath products. These groups are also known as lamplighter groups. The wreath product $K \wr H$, i.e., the lamplighter group with base-group H and lamp-group K, will be defined precisely below. If we think of the elements of K as representing different colors (possibly countably many different colors), then an element of $K \wr H$ can be viewed as a pair (h, η) where h is an element of H (the position of the lamplighter on the base H) and $\eta = (k_h)_{h\in H} \in K^H$ is a finite configuration of colors on H in the sense that only finitely many $h \in H$ have $k_h \neq e_K$ where e_K is the identity element in K (only finitely many lamps are turned on). This description does not explain the group law on $K \wr H$ but captures the nature of the elements of the wreath product $K \wr H$. The identity element in $K \wr H$ has the lamplighter sitting at e_H and all the lamps turned off ($k_h = e_K$ for all $h \in H$). In one of the simplest instance of this construction, $H = \mathbb{Z}$ (a doubly infinite street) and $K = \mathbb{Z}/2\mathbb{Z}$ (lamps are either off (0) or on (1)).

We are interested in a large collection of random walks on wreath products which can be described collectively as the "switch–walk–switch" walks. See also [13, 18]. Namely, we are given two probability measures, one on H, call it μ, and one on K, call it ν. The measure μ drives a random walk on H which describes the moves of the lamplighter (i.e., the first coordinate, h, in the pair $(h, \eta) \in K \wr H$). The measure ν drives a random walk on K whose basic step is interpreted as "switching" between lamp colors. Based on this input, we construct a probability measure $q = q(\mu, \nu)$ on $K \wr H$ (this measure q is defined precisely later in the paper). The basic step of the walk driven by q can be accurately describes as follows: the lamplighter switches the color of the lamp at its standing position (using ν), takes a step in H (using μ) and switches the color of the lamp at its new position (using ν). These different moves are, in the appropriate sense, made independently of each other hence the name, *switch–walk–switch*. Let us insist on the fact that we will be interested here in cases when the measures μ and ν are not necessarily finitely supported. Now, an elementary argument shows that the probability of return $q^{(n)}(e)$ of the random walk driven by q on $K \wr H$ is given by

$$q^{(n)}(e) = \mathbf{E}\left(\prod_h \nu^{(l_*(2n,h))}(e_K)\mathbf{1}_{\{X_n=e_h\}}\right)$$

where $(X_m)_0^\infty$ is the random walk on H driven by μ and $l_*(n,h)$ is an essentially trivial modification of the number of visits of $(X_m)_0^\infty$ to h up to time n. The expectation is relative to the random walk $(X_m)_0^\infty$ on H, started at e_H. This observation goes back to [18] and is the basis of the analysis developed in [13].

If we set $F(m) = -\log \nu^{(2m)}(e_K)$ then it follows under mild assumptions that

$$\log q^{(n)}(e) \sim \log \mathbf{E}\left(e^{-\sum_h F(l(n,h))}\right). \tag{1.1}$$

In words, the log-asymptotic of the probability of return of a switch-walk-switch random walk on the wreath product $K \wr H$ is given by the appropriate version of the Donsker-Varadhan large deviation theorem for the random walk on the base H driven by μ. The particular functional $\sum_h F(l(n,h)$ that needs to be treated depends on the nature of the lamp-group K and the measure ν. Formula (1.1) is particularly interesting because, in the general context of random walks on groups, precise log-asymptotic of the probability of return are hard to obtain.

Equation (1.1) motivates our desire to extend the Donsker–Varadhan theorem to the operator-stable case and to more general functionals of the occupation times. For instance, in [16], the authors initiated the study of the following class of random walks. Fix a group G and a generating k-tuple $S = (s_1, \dots, s_k)$. Fix also a k-tuple $a = (\alpha_1, \dots, \alpha_k) \in (0,2)^k$ and consider the measure

$$\mu_{S,a}(g) = \frac{1}{k}\sum_i \sum_{n\in\mathbb{Z}} \mathbf{1}_{\{s_i^n\}}(g)\mu_i(n), \quad \mu_i(n) = c_i(1+|n|)^{-1-\alpha_i}. \tag{1.2}$$

In words, this walk takes steps along the (discrete) one parameter groups $\langle s_i \rangle = \{s_i^n, n \in \mathbb{Z}\} \subset G$ and the steps along $\langle s_i \rangle$ are distributed according to a symmetric stable-like power law with exponent α_i. These measures $\mu_{S,a}$ are very natural from an algebraic point of view and one expects that the properties of the associated random walks depend in interesting way on the structure of the group G, the generating k-tuple S and the choice of the k-dimensional parameter a.

The following theorem illustrates the above discussion and the type of results we are interested in.

Theorem 1.1. *On the square lattice $\mathbb{Z}^d$, let $s_1, \dots, s_d$ be the canonical generators. Fix $a = (\alpha_i)_1^d \in (0,2)^d$ and let $\mu_a = \mu_{S,a}$ with $S = (s_1, \dots, s_d)$. Define $\alpha = \alpha(a) \in (0,2)$ by*

$$\frac{1}{\alpha} = \frac{1}{d}\sum_1^d \frac{1}{\alpha_i}.$$

Let K be a group of polynomial volume growth equipped with a symmetric probability measure ν that is finitely supported with generating support containing e_K. Let $q = q(\mu_a, \nu)$ be the associated switch–walk–switch measure on the wreath product $K \wr \mathbb{Z}^d$. Then there exists a constant $c = c(K, \nu, a, d)$ such that

$$-\log q^{(n)}(e) \sim c \begin{cases} n^{d/(d+\alpha)} & \text{if } K \text{ is finite} \\ n^{d/(d+\alpha)}(\log n)^{\alpha/(d+\alpha)} & \text{otherwise.} \end{cases}$$

Extensions of this result to base-groups H other than $\mathbb{Z}^d$ is an interesting and challenging problem. Partial results in this direction are developed in the companion paper [15] in the context of nilpotent groups.

For any finitely generated group G and any $\alpha \in (0,2)$, [1] introduces a non-increasing function

$$\widetilde{\Phi}_{G,\rho_\alpha} : \mathbb{N} \ni n \to \widetilde{\Phi}_{G,\rho_\alpha}(n) \in (0,\infty)$$

which, by definition, provides the best possible lower bound

$$\exists\, c > 0,\ N \in \mathbb{N},\ \forall\, n,\ \ \mu^{(2Nn)}(e) \geq c\widetilde{\Phi}_{G,\rho_\alpha}(n),$$

valid for every measure μ on G satisfying the weak-α-moment condition

$$W(\rho_\alpha, \mu) = \sup_{s>0}\{s\mu(\{g : \rho_\alpha(g) > s\})\} < \infty.$$

Here $|g|$ is the word-length of G with respect to some fixed finite symmetric generating set and $\rho_\alpha(g) = (1+|g|)^\alpha$. For instance, it is well know and easy to see that

$$\widetilde{\Phi}_{\mathbb{Z}^d,\rho_\alpha}(n) \simeq n^{-d/\alpha}.$$

Here and throughout this paper, we write $f \sim g$ if $\lim f/g = 1$ and $f \simeq g$ if there are constants c_i, $1 \leq i \leq 4$, such that $c_1 f(c_2 t) \leq g(t) \leq c_3 f(c_4 t)$ on the relevant real interval or on $\mathbb{N}$. We use $\simeq$ only when at least one of the functions f, g is monotone (or roughly monotone).

The main results of the present work allow us to complement some of the lower bounds proved in [1] for $\widetilde{\Phi}_{G,\rho_\alpha}$ with matching upper bounds (note that upper bounds on $\widetilde{\Phi}_{G,\rho_\alpha}$ are proved by exhibiting a measure with finite weak-α-moment and the appropriate return probability behavior).

Theorem 1.2. *Fix $\alpha \in (0,2)$. Let G be the group $K \wr \mathbb{Z}^d$.*

1. *Assume that K is finite. Then*

$$\log \widetilde{\Phi}_{G,\rho_\alpha}(n) \simeq -n^{d/(d+\alpha)}.$$

2. *Assume that K has polynomial volume growth. Then*

$$\log \widetilde{\Phi}_{G,\rho_\alpha}(n) \simeq -n^{d/(d+\alpha)}(\log n)^{\alpha/(d+\alpha)}.$$

3. *Assume that K is polycyclic with exponential volume growth. Then*

$$\log \widetilde{\Phi}_{G,\rho_\alpha}(n) \simeq -n^{(d+1)/(d+1+\alpha)}.$$

Remark 1.3. The blower bounds are from [1]. The upper bound in the first statement is already in [1] since it is based on the classical large deviation result in [7]. The upper bounds in Statements 2 and 3 make use of the extensions of [7] in the spirit of [2] developed here.

Iterated applications of this technique gives the following Theorem.

Theorem 1.4. *Fix $\alpha \in (0,2)$ and integers $d_1, \dots, d_r$. Given a group K, let*

$$G = (\cdots(K \wr \mathbb{Z}^{d_1}) \wr \cdots) \wr Z^{d_r} \text{ and } d = \sum_1^r d_i.$$

1. *Assume that K is finite. Then*

$$-\log \widetilde{\Phi}_{G,\rho_\alpha}(n) \simeq n^{d/(d+\alpha)}.$$

2. *Assume that K has polynomial volume growth. Then*

$$-\log \widetilde{\Phi}_{G,\rho_\alpha}(n) \simeq n^{d/(d+\alpha)}(\log n)^{\alpha/(d+\alpha)}.$$

2 Operator-stable laws

This section describes natural extensions and variations around the classical large deviation theorem of Donsker and Varadhan for the Laplace transform of the number of visited points when the limit of the underlying random walk is operator-stable. The proofs are rather technical but are based on exiting results and well-known arguments developed in variety of sources. They are postponed to a later section where we will give a complete outline but omit details that follow closely the exiting literature. The companion paper [15] discuss the extension of these results and techniques to the case when $\mathbb{Z}^d$ is replaced by a finitely generated nilpotent groups, a context that raises a number of interesting challenges.

For $\alpha \in (0,2)$, the rotationally symmetric α-stable law with density f_α on $\mathbb{R}^d$ is the probability distribution whose Fourier transform is $e^{-|\xi|^\alpha}$. It generates a convolution semigroup with density f_α^t which satisfies $f_\alpha^t(x) = t^{-d/\alpha} f_\alpha \circ \delta_{1/t}^\alpha$ where δ_t^α is the isotropic dilation $\delta_t^\alpha(x) = t^\alpha x$, $x \in \mathbb{R}^d$, $t > 0$.

In the next section, we briefly review the definition of operator-stable laws. In this definition, the role of the isotropic dilations is played by more general one-parameter groups of transformations t^E where E is an endomorphism of the underlying vector space. Given a borel measure μ, we let $t^E(\mu)$ be the Borel measure defined by $t^E(\mu)(A) = \mu(t^{-E}(A))$.

2.1 Operator-stable laws

Let $\mathbb{V}$ be a finite dimensional vector space equipped with the Euclidean scalar product $\langle \cdot, \cdot \rangle$. Let $\mathcal{M}^1(\mathbb{V})$ denote the set of probability measures on $\mathbb{V}$. Given $\mu \in \mathcal{M}^1$, let $\hat{\mu} = e^{-\psi}$ denotes its Fourier transform. Let $\mathcal{ID}(\mathbb{V})$ denotes the set of infinitely divisible laws on $\mathbb{V}$. Throughout this section, we use notation compatible with [11]. Recall that if $\mu \in \mathcal{ID}(\mathbb{V})$ with Fourier transform $e^{-\psi}$ then $e^{-t\psi}$ is the Fourier transform of a probability measure μ^t and $(\mu^t)_{t \geq 0}$ is a continuous convolution semigroup of measure (uniquely determined by μ). Of course, for $\mu \in \mathcal{ID}(\mathbb{V})$, the function ψ admits a Levy-Khinchine representation so that $y \mapsto \psi(y)$ is the sum of three terms, namely, the drift term $-i\langle c, y \rangle$ with

$c \in \mathbb{V}$, the Gaussian term $\frac{1}{2}\langle Qy, y\rangle$, where $Q \in \mathrm{End}^+(\mathbb{V})$, and the generalized Poisson term

$$-\int_{\mathbb{V}^*}\left(e^{i\langle x,y\rangle}-1-\frac{i\langle x,y\rangle}{1+\|x\|^2}\right)W(dx)$$

where W is a Levy measure. Following [11], we call the triple (c, Q, W) the L-K triple of μ (this triple is uniquely determined by μ). We will be interested in the symmetric case where $c = 0$ and $W(x) = W(-x)$. In this case, the Poisson term of the Levy-Khinchine formula equals

$$\int_{\mathbb{V}^*}(1-\cos(\langle x,y\rangle)W(dx).$$

Definition 2.1 (Definition 1.3.11 [11])**.** A law $\eta \in \mathcal{ID}(\mathbb{V})$ is said to be operator-stable if there exist $E \in \mathrm{End}(\mathbb{V})$ and a mapping $a : \mathbb{R}_+^\times \to \mathbb{V}$ such that

$$t^E(\eta) * \delta_{a(t)} = \eta^t,$$

for all $t \in \mathbb{R}_+^\times$. In this case, E is called an exponent of η. Let $\mathrm{EXP}(\eta)$ denote the set of exponents of η. If $a \equiv 0$, η is said to be strictly operator-stable.

Comparing the L-K triple of η under the transformation t^E with the L-K triple of η^t, we have the following characterization of operator-stable distributions.

Lemma 2.2 (Lemma 1.3.12 [11])**.** *Let $\eta \in \mathcal{ID}(\mathbb{V})$ with L-K triple (c, Q, W), and let $E \in End(\mathbb{V})$. Then the following assertions are equivalent:*

1. *η is operator-stable with exponent E.*
2. *$t^E Q t^{E^*} = t \cdot Q$ and $t^E(W) = t \cdot W$ for all $t \in \mathbb{R}_+^\times$.*
3. *$EQ + QE^* = Q$ and $t^E(W) = t \cdot W$ for all $t \in \mathbb{R}_+^\times$.*

One can always split an operator-stable law into a Gaussian part and a generalized Poisson part. Given $Q \in \mathrm{End}^+(\mathbb{V})$, we let η_Q be the (Gaussian) law associated with the triple $(0, Q, 0)$ and $e(W)$ the (generalized-Poisson) law associated with $(0, 0, W)$.

Theorem 2.3 (Theorem 1.3.14 [11] , splitting of operator-stable laws)**.** *Let $\eta \in \mathcal{ID}(\mathbb{V})$ with L-K triple (c, Q, W) be operator-stable with $E \in \mathrm{EXP}(\eta)$.*

1. *$\mathbb{V}_g := \mathrm{Im}(Q)$ is an E-invariant linear subspace of $\mathbb{V}$. Let E_g denote the restriction of E to $\mathbb{V}_g$. Then $E_g \in GL(\mathbb{V}_g)$ and every $z \in \mathrm{Spec}(E_g)$ is simple,* $\mathrm{Re} z = \frac{1}{2}$.
2. *$\mathbb{V}_p := \mathrm{supp}(W)$ is an E-invariant linear subspace of $\mathbb{V}$. Let E_p denote the restriction of E to $\mathbb{V}_p$. Then $E_p \in GL(\mathbb{V}_p)$ and every $z \in \mathrm{Spec}(E_p)$ satisfies* $\mathrm{Re} z > \frac{1}{2}$.
3. *The symmetric Gaussian law η_Q is strictly operator stable with exponent E and* $\mathrm{supp}(\eta_Q) = \mathbb{V}_g$.

4. *The generalized Poisson law* $e(W)$ *is operator stable with exponent* E, *and* $\mathrm{supp}(e(W)) = \mathbb{V}_p$.

5. $\mathbb{V}_g \cap \mathbb{V}_p = \{0\}$, *and* $\eta = \delta_c * \eta_Q * e(W)$.

Now we restrict our attention to symmetric operator stable laws (so that $c = 0, W(dx) = W(-dx)$). Choose an orthonormal basis $\{e_i\}$ on $\mathbb{V}$ with respect to inner product $<,>$. The generating functional A of $(\eta^t)_{t\geq 0}$ (see [11, 1.3.16]) is given for $f \in C^2(\mathbb{V})$ by

$$\begin{aligned} < A, f > &= \frac{1}{2}\sum q_{ij} \cdot \frac{\partial^2}{\partial x_i \partial x_j} f(0) \\ &+ \int_{\mathbb{V}^\times} \left[f(x) - f(0) - \sum \frac{\partial}{\partial x_i} f(0) \cdot \frac{x_i}{1 + \|x\|^2}) \right] W(dx). \end{aligned}$$

One can also write down the Dirichlet form of the continuous convolution semi-group $(\eta^t)_{t\geq 0}$ as

$$\begin{aligned} \mathcal{E}_\eta(f, g) &= \frac{1}{2}\int_{\mathbb{R}^d} \sum q_{ij} \cdot \frac{\partial f}{\partial x_i}(x) \frac{\partial g}{\partial x_j} dx \\ &+ \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (f(x+y) - f(x))(g(x+y) - g(x)) W(dy) dx, \\ \mathcal{D}(\mathcal{E}_\eta) &= \{f \in L^2(\mathbb{V}) : \mathcal{E}_\eta(f, f) < \infty\}. \end{aligned}$$

From the splitting theorem it follows that (q_{ij}) is semi-positive definite, the subspace where it is positive definite is the Gaussian part $\mathbb{V}_g$, and the support of the Levy measure W is the stable part $\mathbb{V}_p$.

Example 2.1. One can construct radial operator-stable laws with respect to non-isotropic homogeneous norms. On $\mathbb{V} = \mathbb{R}^d$, let E be a $d \times d$ diagonal matrix with diagonal entries $a_i \in (\frac{1}{2}, \infty)$. We may assume that $a_1 = \min_{1\leq i\leq d} a_i$. Since

$$t^E = \begin{pmatrix} t^{a_1} & 0 & \dots & 0 \\ 0 & t^{a_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t^{a_d} \end{pmatrix},$$

we can think of t^E as dilations scaling differently in different coordinates. The following norm was considered in [12]. Let $A = \{x : \|x\| < 1\}$ be the open Euclidean unit ball, define

$$\|x\|_{*,E} := \inf\{t : t^{-a_1^{-1}E} x \in A\}.$$

From Theorem 1 in [12], $\|\cdot\|_{*,E}$ is a sub-additive homogeneous norm. Set

$$W(dx) = \frac{c}{\|x\|_{*,E}^{a_1^{-1} + \mathrm{tr}(a_1^{-1}E)}}.$$

Clearly, $t^E(W) = tW$ for all $t \in \mathbb{R}_+^\times$. Let η be the generalized Poisson law with L-K triple $(0, 0, W)$. Then η is operator-stable with exponent E. Note that the assumption $a_1 > \frac{1}{2}$ is needed so that W is a Lévy measure.

2.2 Domain of operator-attraction

For full probability laws, the class of operator-stable laws coincides with limit distributions of normalized sums of i.i.d. random variables and convergence in law of normalized sums can be characterized in terms of convergence of Fourier transforms or convergence of generators as in Trotter's theorem. More precisely, we have the following equivalent characterizations of convergence.

Theorem 2.4 ([11, Theorem 1.6.12 and Corollary 1.6.18])**.** *Let $\mu, \eta \in \mathcal{M}^1(\mathbb{V})$ with $\eta \in \mathcal{ID}(\mathbb{V})$ and $\hat{\eta} = e^{-i\psi}$. Let $T_n \in GL(\mathbb{V})$ and set $\mu_n = T_n\mu$, $\psi_n := \widehat{\mu_n} - 1$, $A_n := \mu_n - \delta_0$. The following properties are equivalent:*

1. *$\mu_n^{(n)} \Longrightarrow \eta$.*
2. *$\mu_n^{(\lfloor nt \rfloor)} \Longrightarrow \eta^t$, uniformly in t over compact subsets of $[0, \infty)$.*
3. *$n\psi_n \to \psi$ uniformly on compact subsets.*
4. *$< nA_n, f > \to < A, f >$ for any $f \in C^2(\mathbb{V})$.*

Next, we introduce the definition of strict domain of operator-attraction.

Definition 2.5 (Definition 1.6.3 [11])**.** Let $\eta \in \mathcal{M}^1(\mathbb{V})$. Then the strict domain of operator-attraction $DOA_s(\eta)$ of η consists of all $\mu \in \mathcal{M}^1(\mathbb{V})$ for which there exists a sequence T_n in $GL(\mathbb{V})$ such that

$$\eta = \lim_{n \to \infty} T_n(\mu^{(n)}).$$

Remark 2.6. With this definition, $\mathrm{DOA}_s(\eta) \neq \varnothing$ is equivalent to saying η can be obtained as the limiting distribution of convolution powers of some μ after normalization (but without re-centering). The word "strict" refers to the absence of re-centering. When T_n can be taken as the isotropic matrix $b_n Id$, $b_n \in \mathbb{R}_+$, this agrees with the definition of the strict domain of attraction.

Definition 2.7 (Definition 1.10.1 [11])**.** Let $\eta \in \mathcal{M}^1(\mathbb{V})$ be operator-stable. Then its strict domain of normal operator-attraction $\mathrm{DNOA}_s(\eta)$ consists of all $\mu \in \mathcal{M}^1(\mathbb{V})$ such that

$$\eta = \lim_{n \to \infty} n^{-E}(\mu^{(n)})$$

for some $E \in \mathrm{EXP}(\eta)$.

Example 2.2. Let U (resp. V) be a random variable on $\mathbb{Z}$ in the domain of normal attraction of the α (resp. β) symmetric-stable law ν_α (ν_β resp.) on $\mathbb{R}$. Then the measure η on $\mathbb{R}^2$

$$\eta(dx, dy) = \frac{1}{2}\nu_\alpha(dx)\mathbf{1}_{\{y=0\}} + \frac{1}{2}\nu_\beta(dy)\mathbf{1}_{\{x=0\}}$$

is operator stable with exponent $E = \begin{pmatrix} \frac{1}{\alpha} & 0 \\ 0 & \frac{1}{\beta} \end{pmatrix}$. It is clear that the law of $(U, V)^T$) is in $\mathrm{DNOA}_s(\eta)$. Set

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix},$$

and let μ denote the distribution of $(X, Y)^T$. In order to obtain convergence of $\mu^{(n)}$ we need to rotate back by a rotation of angle θ then normalize component-wise. That is, setting

$$T_n = \begin{pmatrix} n^{-\frac{1}{\alpha}} & 0 \\ 0 & n^{-\frac{1}{\beta}} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

we have $\eta = \lim_{n\to\infty} T_n(\mu^{(n)})$. So, in this case, $\mu \in \mathrm{DOA}_s(\eta)$ but does not belong to $\mathrm{DNOA}_s(\eta)$.

Notation 1. A measure $\mu \in \mathcal{M}^1(\mathbb{V})$ is said to be adapted if μ is not supported by a proper linear subspace of $\mathbb{V}$. Let $\mathcal{M}^1_a(\mathbb{V})$ denote the set of adapted probability measures on $\mathbb{V}$.

The theorem below is a characterization of strictly operator-stable laws as those adapted distributions whose domain of strict operator-attraction is non-empty.

Theorem 2.8 (Theorem 1.6.4 [11])**.** *For $\eta \in \mathcal{M}^1_a(\mathbb{V})$ the following assertions are equivalent:*

1. *η is strictly operator-stable.*
2. *$\eta \in \mathrm{DOA}_s(\eta)$.*
3. *$\mathrm{DOA}_s(\eta) \neq \varnothing$.*

For $\mu \in \mathrm{DOA}_s(\eta)$, the choice of normalization sequence T_n is in general not unique. In particular, we can adjust T_n using the symmetries of the limiting distribution η and the convergence still holds.

Definition 2.9 (Definition 1.2.8. [11])**.** Let $\eta \in \mathcal{M}^1(\mathbb{V})$ be non-degenerate. Let $\mathrm{Sym}(\eta)$ be the set of all $A \in GL(\mathbb{V})$ such that there exists some $a \in \mathbb{V}$ such that $A(\eta) * \delta_a = \eta$. The group $\mathrm{Sym}(\eta)$ is called the symmetry group of η. It is a closed subgroup of $GL(\mathbb{V})$. The invariance group $\mathrm{Inv}(\eta)$ is the set of all $A \in GL(\mathbb{V})$ such that $A(\eta) = \eta$. The group $\mathrm{Inv}(\eta)$ is a closed subgroup of $\mathrm{Sym}(\eta)$.

The following technical result is important for our purpose. It says that we can always adjust the normalization sequence by elements in $\mathrm{Inv}(\eta)$, so that the new normalization sequence has nice regular variation properties.

Theorem 2.10 (Theorem 1.10.19 [11]). *Suppose μ is in the strict domain of attraction of a full operator stable law η, that is, there exists a sequence of invertible matrices $B_n \in GL(\mathbb{V})$ such that*

$$B_n^{-1}\mu^{(n)} \Longrightarrow \eta.$$

Then there exists a modified normalization sequence $\{B_n' = B_n S_n\}$, $S_n \in \mathrm{Inv}(\eta)$, hence still fulfilling

$$(B_n')^{-1}\mu^{(n)} \Longrightarrow \eta,$$

with the property that $\{B_n'\}$ has regular variation in the sense that

$$B_n'(B_{\lfloor nt \rfloor}')^{-1} \to t^{-E},$$

where the convergence is uniform in t on compact subsets of $\mathbb{R}_+^\times$.

2.3 Two more examples on $\mathbb{Z}^2$

In this subsection we discuss two examples on $\mathbb{Z}^2$ that are in the strict domain of operator-attraction of some operator-stable laws. For later use, we include the additional requirement that the inverse of the normalization sequence preserve the lattice $\mathbb{Z}^2$.

Example 2.3. Take two linearly independent unit vectors u_1 and u_2 in $\mathbb{R}^2$. Consider the generalized Poisson law η with Lévy measure W supported on one-dimensional subspaces $\mathbb{R}u_1$ and $\mathbb{R}u_2$,

$$W(dx) = \frac{\lambda_1}{|t|^{\alpha+1}}\mathbf{1}_{\{x=tu_1\}}dt + \frac{\lambda_2}{|s|^{\beta+1}}\mathbf{1}_{\{x=su_2\}}ds,$$

where λ_1, λ_2 are positive constants, $\alpha, \beta \in (0,2)$. Write $P = (u_1, u_2)$, then η is operator-stable with exponent E

$$E = P\begin{pmatrix} \frac{1}{\alpha} & 0 \\ 0 & \frac{1}{\beta} \end{pmatrix}P^{-1}.$$

Set $B_n = \left\lfloor P\begin{pmatrix} n^{\frac{1}{\alpha}} & 0 \\ 0 & n^{\frac{1}{\beta}} \end{pmatrix}P^{-1}\right\rfloor$ where $\lfloor\cdot\rfloor$ means take integer parts of each matrix entry. Then $n^E - B_n$ is a matrix with entries in $[0,1)$ and it follows that $B_n \cdot n^{-E} \to I$. Let $p_i(x)$, $i = 1, 2$, be coordinates of x with respect in $(u_1, u_2$, that is,

$$\begin{pmatrix} p_1(x) \\ p_2(x) \end{pmatrix} = P^{-1}x.$$

Fix two nonnegative functions ϕ_1, ϕ_2 on $[0, \infty)$ with $\lim_{s\to\infty} \phi_i(s) = 0$. Let φ_1 be a nonnegative function whose support is contained in

$$\{x \in \mathbb{Z}^2 : |p_2(x)|^\beta \le |p_1(x)|^\alpha \phi_1(|p_1(x)|)\}$$

and

$$\sum_{x\in\mathbb{Z}^2, p_1(x)\in[n,n+1)} \varphi_1(p_1(x), p_2(x)) \sim \frac{1}{|n|^{\alpha+1}}.$$

Similarly, let φ_2 be a nonnegative function whose support is contained in

$$\{x \in \mathbb{Z}^2 : |p_1(x)|^\alpha \le |p_2(x)|^\beta \phi_2(|p_2(x)|)\}$$

and

$$\sum_{x:p_2(x)\in[n,n+1)} \varphi_2(p_1(x), p_2(x)) \sim \frac{1}{|n|^{\beta+1}}.$$

Now, consider the measure μ supported on $\mathbb{Z}^2$ with

$$\mu(x) = \lambda_1\varphi_1(p_1(x), p_2(x)) + \lambda_2\varphi_2(p_1(x), p_2(x)),$$

Then we can check that

$$nB_n^{-1}\mu \to W$$

weakly. From the convergence theorem, Theorem 2.4, we conclude that $B_n^{-1}\mu^{(n)} \Longrightarrow \eta$.

Example 2.4. Take u_1, u_2 and E be as in the previous example. Let S^1 denote the Euclidean unit circle. Let Γ be the union of the two arcs $[0, \frac{\pi}{2}]$ and $[\pi, \frac{3\pi}{2}]$ on S^1. Define the Lévy measure W by

$$W = \int_\Gamma \int_0^\infty \frac{1}{t^2}\delta_{t^E P\sigma} dt d\sigma.$$

Then W is supported in the cone $\{x \in \mathbb{R}^2 : p_1(x)p_2(x) > 0\}$. Define a discrete approximation μ of W supported on $\mathbb{Z}^2$ by setting

$$\mu(x) = W([x_1, x_1 + 1) \times [x_2, x_2 + 1)).$$

One can check that $nB_n^{-1}\mu \to W$ with $B_n = \lfloor n^E \rfloor$ and it follows that

$$B_n^{-1}\mu^{(n)} \Longrightarrow \eta.$$

3 Functionals of the occupation time vector

Given a probability measure μ on the lattice $\mathbb{Z}^d$, let $(X_i)_0^\infty$ be the associated random walk. Let $(l(n,x))_{x\in\mathbb{Z}^d}$ be the occupation time vector at time n where $l(n,x) = \#\{k \in \{0,\dots n\} : X_k = x\}$. Let $F : [0,\infty) \to [0,\infty)$.

In this section we introduce basic natural hypotheses on μ and F under which we can derive the log-asymptotic behavior of

$$\mathbf{E}\left(e^{-\sum_{x\in\mathbb{Z}^d} F(l(n,x))}\right).$$

Definition 3.1 (Convergence assumption)**.** We say that μ satisfies the convergence assumption (C-B_n) if there exists a sequence of invertible matrices $B_n \in \mathbb{Z}^{d\times d}$ and a probability distribution η such that

$$B_n^{-1}\mu^{(n)} \Longrightarrow \eta. \tag{C-B_n}$$

Remark 3.2*.* Note that (C-B_n) requires the matrices B_n to have integer entries so that $B_n\mathbb{Z}^d \subset \mathbb{Z}^d$. Note also that the distribution η is strictly operator-stable.

Under the convergence assumption (C-B_n), [10] provides a local limit theorem that plays an important role in the proof of the uniform large deviation principle.

Theorem 3.3 (Theorem 6.4 [10])**.** *Suppose μ is in the domain of attraction of a symmetric, adapted strictly operator-stable law η on $\mathbb{R}^d$ with density g, that is, there exists a sequence of invertible matrices B_n such that*

$$B_n^{-1}\mu^{(n)} \Longrightarrow \eta.$$

Then

$$\limsup_{n\to\infty} \sup_{x\in\mathbb{Z}^d} |\det B_n| \left|\mu^{(n)}(x) - \left|\det B_n^{-1}\right| g(B_n^{-1}x)\right| = 0.$$

Remark 3.4*.* Note that the density g of an operator-stable law is always smooth. In [7] it is essentially proved, although not stated explicitly, that given the local limit theorem, the scaled occupation time measures satisfy a uniform large deviation principle in L_1. We will state and outline the proof of the large deviation principles later in this paper.

Remark 3.5*.* It is somewhat surprising that, in this case, the "weak limit assumption" always implies the local limit theorem. The proof in [10] relies on the Fourier transform. On the Heisenberg group, there are measures that converges to a (Heisenberg group) Gaussian law, but do not satisfy the local limit theorem.

Next we introduce a scaling assumption regarding the function F. It is the operator-stable analog of the scaling assumption in [2].

Definition 3.6 (Scaling assumption)**.** Let B_n be as in condition (C-B_n). We say that a function $F : [0,\infty) \to [0,\infty)$ satisfies the scaling assumption (S-B_n-a_n) if F is concave, sub-additive, increasing with $F(0) = 0$ and there exist a non-decreasing sequence $n \to a_n \in \mathbb{N}$ and a limiting function $\widetilde{F} : [0,\infty) \to [0,\infty)$, $\widetilde{F}$ not identically zero, such that for $y > 0$,

$$\lim_{n\to\infty} \frac{a_n \det(B_{a_n})}{n} F\left(\frac{n}{\det(B_{a_n})} y\right) = \widetilde{F}(y), \tag{S-B_n-a_n}$$

uniformly over compact sets in $(0,\infty)$.

The following technical proposition is crucial. It is analogous to [2, Proposition 1.1]. The proof is given in the Appendix.

Proposition 3.7. *Assume the convergence assumption* (C-B_n) *and the scaling assumption* (S-B_n-a_n) *as above. Then there exists* $\gamma \in [0,1]$ *such that*

$$\widetilde{F}(y) = \widetilde{F}(1) y^\gamma, \ \ y > 0,$$

Moreover, there exists $\kappa > 0$ *such that*

$$\lim_{n\to\infty} \frac{a_{\lfloor \lambda n \rfloor}}{a_n} = \lambda^\kappa \text{ for all } \lambda \in \mathbb{R}^+ \text{ and } \lim_{n\to\infty} \frac{\log a_n}{\log n} = \kappa.$$

Definition 3.8. Following [2], given μ satisfying (C-B_n) and a function $F : [0,\infty) \to [0,\infty)$, we say that the pair $(F, (B_n))$ is in the γ-class, if there is a sequence a_n such that the scaling assumption (S-B_n-a_n) is satisfied, and the limiting function $\widetilde{F}$ is homogeneous with exponent γ.

Theorem 3.9. *Fix a symmetric probability measure* μ *on* $\mathbb{Z}^d$ *and a function* $F : [0,\infty) \to [0,\infty)$*. Under the convergence assumption (C-*B_n*) and the scaling assumption (S-*B_n*-*a_n*), there exists a constant* $k(\eta, \widetilde{F}) \in (0,\infty)$ *such that*

$$\lim_{n\to\infty} \frac{a_n}{n} \log \mathbf{E}\left(e^{-\sum_{x\in\mathbb{Z}^d} F(l(n,x))}\right) = -k\left(\eta, \widetilde{F}\right). \tag{3.1}$$

Further, for any $\epsilon > 0$ *small enough there is* $R > 1$ *such that*

$$\lim_{n\to\infty} \frac{a_n}{n} \log \mathbf{E}\left(e^{-\sum_{x\in\mathbb{Z}^d} F(l(n,x))} \mathbf{1}_{B(Ra_n)}(X_n)\right) \geq -(1+\epsilon) k\left(\eta, \widetilde{F}\right). \tag{3.2}$$

Example 3.1. Assume that $\mu \in \mathrm{DNOA}_s(\eta)$, $B_n = \lfloor n^E \rfloor$, $\mathrm{tr}(E) = \tau$ and $F(y) = y^\gamma \ell(y)$ where $\gamma \in [0,1]$ and ℓ is a slow varying function (at infinity) such that $\ell(t^a \ell(t)^b) \sim c(a)\ell(t)$ for any $a > 0$ and $b \in \mathbb{R}$. (e.g., $\ell(t) = (\log t)^\beta$, $\beta \in \mathbb{R}$). Then $\widetilde{F}(y) = cy^\gamma$ and a_n is determined by solving

$$a_n^{1+\tau(1-\gamma)} \ell(n a_n^{-\tau}) = n^{1-\gamma},$$

that is

$$a_n \sim c \left(\frac{n^{1-\gamma}}{\ell(n)}\right)^{1/(1+\tau(1-\gamma))}.$$

In this case the theorem yields the existence of a constant $k \in (0,\infty)$ such that

$$\log \mathbf{E}\left(\exp\left(-\sum_{x\in\mathbb{Z}^d} F(l(n,x))\right)\right) \sim -k\left(n^{\gamma+\tau(1-\gamma)}\ell(n)\right)^{1/(1+\tau(1-\gamma))}.$$

In the case where $F(y) = 1-\delta_0(y)$, i.e., when dealing with the number of visited sites, we have $\gamma = 0$, $\ell \equiv 1$ and the result simplifies to

$$\log \mathbf{E}\left(\exp\left(-\sum_{x\in\mathbb{Z}^d} F(l(n,x))\right)\right) \sim -kn^{\tau/(1+\tau)}.$$

If η is the (classical) symmetric α-stable law, $\tau = d/\alpha$ and $\tau/(1+\tau) = d/(d+\alpha)$ as in the original result of Donsker and Varadhan.

4 Applications to random walks on groups

This section applies the large deviation asymptotics of Theorem 3.9 to obtain precise information about the decay of the return probability of random walks on wreath products with base $\mathbb{Z}^d$. We treat certain classes of random walks with unbounded support on the base and we allow a large class of lamp groups.

4.1 Random walks on wreath products

First we briefly review definition of wreath products and a special type of random walks on them. Our notation follows [13]. Let H, K be two finitely generated groups. Denote the identity element of K by e_K and identity element of H by e_H Let K_H denote the direct sum:

$$K_H = \sum_{h\in H} K_h.$$

The elements of K_H are functions $f : H \to K$, $h \mapsto f(h) = k_h$, which have finite support in the sense that $\{h \in H : f(h) = k_h \neq e_K\}$ is finite. Multiplication on K_H is simply coordinate-wise multiplication. The identity element of K_H is the constant function $\boldsymbol{e}_K : h \mapsto e_K$ which, abusing notation, we denote by e_K. The group H acts on K_H by translation:

$$\tau_h f(h') = f(h^{-1}h'), \quad h, h' \in H.$$

The wreath product $K \wr H$ is defined to be semidirect product

$$K \wr H = K_H \rtimes_\tau H,$$

$$(f,h)(f',h') = (f\cdot\tau_h f', hh').$$

In the lamplighter interpretation of wreath products, H corresponds to the base on which the lamplighter lives and K corresponds to the lamp. We embed K and H naturally in $K \wr H$ via the injective homomorphisms

$$\begin{array}{lll} k & \longmapsto & \underline{k} = (\boldsymbol{k}_{e_H}, e_H), \;\; \boldsymbol{k}_{e_H}(e_H) = k, \; \boldsymbol{k}_{e_H}(h) = e_K \text{ if } h \neq e_H \\ h & \longmapsto & \underline{h} = (\boldsymbol{e}_K, h). \end{array}$$

Let μ and ν be probability measures on H and K respectively. Through the embedding, μ and ν can be viewed as probability measures on $K \wr H$. Consider the measure

$$q = \nu * \mu * \nu$$

on $K \wr H$. This is the switch-walk-switch measure on $K \wr H$ with switch-measure ν and walk-measure μ.

Let (X_i) be the random walk on H driven by μ, and let $l(n,h)$ denote the number of visits to h in the first n steps:

$$l(n,h) = \#\{i : 0 \leq i \leq n, \; X_i = h\}.$$

Set also

$$l_*^g(n,h) = \begin{cases} l(n,h) & \text{if } h \notin \{e_H, g\} \\ l(n,e_H) - 1/2 & \text{if } h = g \\ l(n,e_H) - 1 & \text{if } h = e_H. \end{cases}$$

From [13], probability that the random walk on $K \wr H$ driven by q is at $(h,g) \in K \wr H$ at time n is given by

$$q^{(n)}((f,g)) = \mathbf{E}\left(\prod_{h \in H} \nu^{(2l_*^g(n,h))}(f(h)) \mathbf{1}_{\{X_n = g\}}\right)$$

Note that $\mathbf{E}$ stands for expectation with respect to the random walk $(X_i)_0^\infty$ on H started at e_H.

From now on we assume that ν satisfies $\nu(e_K) = \epsilon > 0$ so that

$$\epsilon \nu^{(n-1)}(e_K) \leq \nu^{(n)}(e_K) \leq \epsilon^{-1} \nu^{(n-1))}(e_K).$$

Write $f \overset{C}{\asymp} g$ if $C^{-1} f \leq g \leq C f$. Under these circumstances, we have

$$q^{(n)}((\boldsymbol{e}_K, g)) \overset{1/\epsilon^3}{\asymp} \mathbf{E}\left(\prod_{h \in H} \nu^{(2l(n,h))}(e_K) \mathbf{1}_{\{X_n = g\}}\right)$$

so that we can essentially ignore the difference between l and l_*.

Set

$$F_K(n) := -\log \nu^{(2n)}(e_K)$$

so that, for any $h \in H$,

$$q^{(n)}((\boldsymbol{e}_K, h)) \simeq \mathbf{E}\left(e^{-\sum_H F_K(l(n,h))} \mathbf{1}_{\{X_n = h\}}\right). \tag{4.1}$$

Definition 4.1 (weak scaling assumption)**.** We say that ν satisfies the upper weak scaling assumption (US-B_n-a_n) if there exist a constant $c_0 > 0$ and a function $F : [0,\infty) \to [0,\infty)$ satisfying (S-B_n-a_n) and such that

$$\forall\, n \in \mathbb{N},\ \ c_0 F(n) \le F_K(n). \qquad \text{(US-}B_n\text{-}a_n\text{))}$$

We say that ν satisfies the lower weak scaling assumption *(LS-B_n-a_n)* if there exist a constant $C_0 < \infty$ and a function $F : [0,\infty) \to [0,\infty)$ satisfying *(-B_n-a_n)* and such that

$$\forall\, n \in \mathbb{N},\ \ F_K(n) \le C_0 F(n) \qquad \text{(LS-}B_n\text{-}a_n\text{))}$$

If F_K satisfies both the upper and lower conditions,

$$\forall\, n \in \mathbb{N},\ \ c_0 F(n) \le F_K(n) \le C_0 F(n) \qquad \text{(WS-}B_n\text{-}a_n\text{)}$$

then we say it satisfies the weak scaling assumption (WS-B_n-a_n).

We can now transfer the large deviation asymptotics to return probability on wreath product $K \wr \mathbb{Z}^d$.

Theorem 4.2. *Let μ be a symmetric probability measure on $\mathbb{Z}^d$ which satisfies the convergence assumption* (C-B_n)*. Let ν be a symmetric probability measure on K which satisfies* (US-B_n-a_n)*, then the measure $q = \nu * \mu * \nu$ on $K \wr \mathbb{Z}^d$ satisfies*

$$\limsup_{n\to\infty} \frac{a_n}{n} \log q^{(n)}(e) \le -k\left(\eta, c_0\widetilde{F}\right).$$

If instead the symmetric measure ν on K satisfies (LS-B_n-a_n)*, then*

$$\liminf_{n\to\infty} \frac{a_{2n}}{2n} \log q^{(2n)}(e) \ge -k\left(\eta, C_0\widetilde{F}\right).$$

Remark 4.3. Roughly speaking, this theorem says the following: Assume we know how to normalize $\mu^{(n)}$ on the base $\mathbb{Z}^d$ via a transformation B_n so that it converges to a limiting distribution η. Assume we know the behavior of the probability of return of the random walk on K driven by ν in the sense that $\log(\nu^{(2n)}(e_K)) \simeq -F(n)$. Then $q^{(n)}(e) \simeq \exp(-\frac{n}{a_n})$, where a_n can be computed from the scaling relation

$$\frac{a_n \det(B_{a_n})}{n} F\left(\frac{n}{\det(B_{a_n})}\right) \simeq 1.$$

Proof. The first statement follows immediately from (3.1) in Theorem 3.9. The second statement is deduced from (3.2) as follows. Since q is symmetric, $q^{(2n)}(e_H) \ge q^{(2n)}(g)$ for any $g \in K \wr \mathbb{Z}^d$. In particular, if $B(r) = B_{\mathbb{Z}^d}(r)$ is the ball of radius r in the lattice $\mathbb{Z}^d$ then, by (4.1),

$$\begin{aligned}
\#B(r) q^{(2n)}(e) &\ge c \sum_{h\in B(r)} q^{(2n)}(\boldsymbol{e}_K, h) \\
&\simeq \mathbf{E}\left(e^{-\sum_H F_K(l(n,h))} \mathbf{1}_{B(r)}(X_{2n})\right).
\end{aligned}$$

Picking $r = Ra_{2n}$ and using the fact that a_n has regular variation of order $\kappa > 0$ (see Proposition 3.7), one easily deduces from (3.2) that

$$\liminf_{n\to\infty} \frac{a_{2n}}{2n} \log q^{(2n)}(e) \geq -k\left(\eta, C_0\widetilde{F}\right).$$

as desired. $\square$

Example 4.1. (See Example 3.1) Let μ be a symmetric probability measure on $\mathbb{Z}^d$, $\mu \in \mathrm{DNOA}_s(\eta)$, $B_n = \lfloor n^E \rfloor$, $\mathrm{tr}(E) = \tau$ (this implies $\tau \geq d/2$). Let $F(y) = y^\gamma \ell(y)$ where $\gamma \in [0,1]$ and ℓ is a slow varying function (at infinity) such that $\ell(t^a \ell(t)^b) \sim c(a)\ell(t)$ for any $a > 0$ and $b \in \mathbb{R}$. (e.g., $\ell(t) = (\log t)^\beta$, $\beta \in \mathbb{R}$).

Let ν be a symmetric probability measure on K with $\nu(e_K) > 0$ and such that

$$\log \nu^{(2n)}(e_K) \simeq -F(n).$$

Then

$$\log q^{(2n)}(e) \simeq -\left(n^{\gamma+\tau(1-\gamma)}\ell(n)\right)^{1/(1+\tau(1-\gamma))}.$$

For a concrete example, let μ be the uniform probability on

$$\{0, \pm s_1, \ldots, \pm s_d\} \subset \mathbb{Z}^d$$

where $s_1, \ldots, s_d$ are the unit vectors generating the square lattice $\mathbb{Z}^d$. Obviously, μ is in domain of normal attraction of the Gaussian measure and $\tau = d/2$. Take $K = \mathbb{Z} \wr \mathbb{Z}$ and ν be the switch-walk-switch measure on $\mathbb{Z} \wr \mathbb{Z}$ where both the switch-measure and walk-measure are simple random walk on $\mathbb{Z}$ with holding. In this case, $F(y) = y^{1/3}(\log y)^{2/3}$ and $\gamma = 1/3$ (see, e.g., [13]). Hence the measure $q = \nu * \mu * \nu$ on $K \wr \mathbb{Z}^d$ satisfies

$$\log q^{(2n)}(e) \simeq -n^{(1+d)/(3+d)}(\log n)^{2/(3+d)}.$$

We note that this result can also be obtained from Erschler's results [8].

Now, we can also consider the case when $\mu_\alpha(x) = c(1+\|x\|)^{-d-\alpha}$, $\alpha \in (0,2)$, $\|x\| = (\sum_1^d |x_i|^2)^{/2}$. In this case $\mu \in \mathrm{DNOA}_s(\eta_\alpha)$ where η_α is the rotationally symmetric α-stable law on $\mathbb{R}^d$ and $\tau = d/\alpha$. If we set $q_\alpha = \nu * \mu_\alpha * \nu$ then we obtain

$$q^{(2n)}(e) \simeq -n^{(1+2d/\alpha)/(3+2d/\alpha)}(\log n)^{2/(3+2d/\alpha)}.$$

Better understanding of the return probability on the lamp-group K leads to more precise asymptotics for $q^{(n)}(e)$.

Theorem 4.4. *Let μ be a symmetric probability measure on $\mathbb{Z}^d$ which satisfies the convergence assumption* (C-B_n). *Let ν be a symmetric probability measure on K. Assume that the function $F_K(n) = -\log \nu^{(2n)}(e_K)$ satisfies the scaling assumption* (S-B_n-a_n). *Then the measure $q = \nu * \mu * \nu$ on $K \wr \mathbb{Z}^d$ satisfies*

$$\limsup_{n\to\infty} \frac{a_n}{n} \log q^{(n)}(e) = -k\left(\eta, \widetilde{F}_K\right).$$

Example 4.2. Referring to the setting of Theorem 4.4, assume that $\nu^{(2n)}(e_K)$ satisfies $\nu^{(2n)}(e_K) \simeq n^{-\theta}$ so that $F_K(n) \sim \theta \log n$. Assume μ is in the domain of normal attraction of η. Let $E \in \mathrm{EXP}(\eta)$ and $B_n = \lfloor n^E \rfloor$. Let $\tau = \mathrm{tr}(E)$ be the trace of E. Solving for $a_n = t$ in the scaling equation

$$\frac{t^{1+\tau}}{n} \log\left(\frac{n}{t^\tau}\right) = 1,$$

yields

$$a_n = t \sim \left(\frac{n}{\log n}\right)^{\frac{1}{1+\tau}}.$$

Then F_K satisfies the scaling assumption

$$\lim_{n\to\infty} \frac{a_n \det(B_{a_n})}{n} F_K\left(\frac{n}{\det(B_{a_n})} y\right) = \theta, \text{ for } y > 0.$$

Hence Theorem 4.4 yields

$$\lim_{n\to\infty} \frac{1}{n^{\frac{\tau}{1+\tau}} (\log n)^{\frac{1}{1+\tau}}} \log q^{(n)}(e) = -k\left(\eta, \widetilde{F}\right),$$

where the limiting function $\widetilde{F}$ is given by $\widetilde{F}(y) = \theta \cdot \mathbf{1}_{\{y>0\}}$.

4.2 Assorted examples

In this section we describe a number of explicit applications of Theorems 4.2 and 4.4.

Example 4.3. Let $\mathbb{Z}^d$ be equipped with the canonical generating d-tuple $S = (s_1, \dots, s_d)$ and fix $a = (\alpha_1, \dots, \alpha_d) \in (0,2)^d$. Consider the probability measure $\mu_a = \mu_{S,a}$ defined in (1.2). In the usual $\mathbb{Z}^d$ notation, μ_a is given by

$$\mu_a(x) = \frac{1}{d} \sum_{i=1}^{d} \sum_{n\in\mathbb{Z}} \frac{c(\alpha_i)}{(1+|n|)^{1+\alpha_i}} \mathbf{1}_{\{n\}}(x_i), \quad x = (x_1, \dots, x_d). \tag{4.2}$$

This measure is quite obviously in the domain of normal operator attraction of $\eta_a = \frac{1}{d} \sum \eta_{i,\alpha_i}$ where, for each i, η_{i,α_i} is a measure on $\mathbb{R}^d$ supported on the i-th coordinate axis and whose restriction to this axis is symmetric and α_i-stable. In particular, the diagonal $d \times d$ matrix E_a with i-th diagonal entry $1/\alpha_i$ is in $\mathrm{EXP}(\eta_a)$. The Dirichlet form $\mathcal{E}_{\eta_a}$ associated to the limit law η_a is best described via Fourier transform as $\mathcal{E}_{\eta_a}(f,f) = \sum_1^d \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 |\xi_i|^{2\alpha_i} d\xi$.

Theorem 4.5. *On $H = \mathbb{Z}^d$, consider the measures μ_a defined above, $a \in (0,2)^d$. Define $\alpha \in (0,2)$ by*

$$\frac{1}{\alpha} = \frac{1}{d} \sum_1^d \frac{1}{\alpha_i}.$$

1. *Let K be a finite group and let ν be the uniform measure on K. On $K \wr H$, let $q_a = \nu * \mu_a * \nu$. Then there exists a constant $k = k(d, a, |K|)$ such that*

$$\log q_a^{(n)}(e) \sim -kn^{d/(d+\alpha)}.$$

2. *Let $K = \mathbb{Z}^D$ and ν be a symmetric probability measure on $\mathbb{Z}^D$ with $\nu(e)$ and which is in the domain of normal attraction of an adapted strictly operator-stable law η. On $K \wr H$, let $q_a = \nu * \mu_a * \nu$. Then there exists a constant $k = k(d, a, D, \nu)$ such that*

$$\log q_a^{(n)}(e) \sim -kn^{d/(d+\alpha)}(\log n)^{\alpha/(d+\alpha)}.$$

Set $H = \mathbb{Z}^d$, $K = \mathbb{Z}^D$, $G = K \wr H$. One natural set of generators of $G = K \wr H$ is obtained by joining the canonical generators of $H = \mathbb{Z}^d$ and $K = \mathbb{Z}^D$ as follows. Let $(s_i^H)_1^d$ and $(s_i^K)_1^D$ be the canonical generators of H and K, respectively. Let $S = (s_i)_1^{d+D}$ be the generating tuple of G given by

$$s_i = (\boldsymbol{e}_K, s_i^H) \text{ for } i \in \{1, \dots, d\} \text{ and } s_i = (\boldsymbol{s}_i^K, e_H) \text{ for } i \in \{d+1, \dots, d+D\}.$$

Of course, $e_K = 0$ in $\mathbb{Z}^D$ and $e_H = 0$ in $\mathbb{Z}^d$. Let

$$a = (\alpha_1, \dots, \alpha_{d+D}) \in (0, 2)^{d+D}$$

be a $(d+D)$-tuple. Let $b = b(a) = (\beta_i)_1^d$ and $c = c(a) = (\gamma_i)_1^D$ with $\beta_i = \alpha_i$, $i = 1, \dots, d$ and $\gamma_i = \alpha_{d+i}$, $i = 1, \dots, D$. Let μ_b^H, μ_c^K be the probability measures on $H = \mathbb{Z}^d, K = \mathbb{Z}^D$, respectively, defined at (4.2). Let q be the switch-walk-switch measure on $G = K \wr H$ given by $q = \mu_c^K * \mu_b^H * \mu_c^K$. The theorem stated above applies and yields

$$\log q^{(n)}(e) \sim -k(d, D, a)n^{d/(d+\beta)}(\log n)^{\beta/(d+\beta)}, \quad \frac{1}{\beta} = \frac{1}{d}\sum_1^d \frac{1}{\beta_i}.$$

For S and a as defined above, let $\mu_{S,a}$ the the probability measure on $G = K \wr H$ defined at (1.2). The Dirichlet forms $\mathcal{E}_{\mu_{S,a}}$ and $\mathcal{E}_q$ associated with the measures $\mu_{S,a}$ and q on G satisfy

$$\mathcal{E}_{\mu_{S,a}} \simeq \mathcal{E}_q.$$

Hence it follows from [14] that

$$\log \mu_{S,a}(e) \simeq -n^{d/(d+\beta)}(\log n)^{\beta/(d+\beta)}$$

where β is as above. Note that β depends only on the first d coordinates of the parameter $a = (\alpha_i)_1^{d+D}$. In this sense, the random walks associated with the collection of the measures $\mu_{S,a}$ when a varies can distinguish among the $d+D$ generators s_i, $1 \le i \le d+D$ of $K \wr H$ between those which come from H and those which come from K.

Example 4.4. Consider the iterated wreath product

$$(\dots(\mathbb{Z}_2 \wr \mathbb{Z}^{d_1}) \wr \mathbb{Z}^{d_2}) \wr \dots) \wr \mathbb{Z}^{d_k}.$$

Note that $\wr$ is not associative so that this iterated wreath product is different from the iterated wreath product $\mathbb{Z}_2 \wr (\cdots \wr (\mathbb{Z} \wr \mathbb{Z}) \dots)$ considered in [8]. Here we are iterating the lamps while in [8] the base is iterated.

For each $i = 1, \dots, k$, fix $\alpha_i \in (0, 2)$ and a probability measure μ_i on $\mathbb{Z}^{d_i}$ which is symmetric, satisfies $\mu_i(0) > 0$ and is in the domain of normal attraction of the rotationally α_i-stable law η_i on $\mathbb{R}^{d_i}$. Let q_0 be the uniform measure on $\mathbb{Z}_2 = \{0, 1\}$. Iteratively, define the switch-walk-switch probability measure

$$q_i = q_{i-1} * \mu_i * q_{i-1}$$

on $(\dots(\mathbb{Z}_2 \wr \mathbb{Z}^{d_1}) \wr \mathbb{Z}^{d_2}) \wr \dots) \wr \mathbb{Z}^{d_i}$.

Applying Corollary 4.4 iteratively, we obtain

$$\lim_{n\to\infty} n^{-\frac{d_1/\alpha_1+\dots+d_k/\alpha_k}{1+d_1/\alpha_1+\dots+d_k/\alpha_k}} \log q_k^{(n)}(e) = -c_k$$

where the constant c_k can be obtained as follows. The constant c_1 is given by [7] whereas, for $2 \le i \le k$ and referring to (3.1)-(3.2), $c_i = k\left(\upsilon_i, \widetilde{F}_i\right)$ where

$$\widetilde{F}_i(y) = c_{i-1} y^{\frac{d_1/\alpha_1+\dots+d_{i-1}/\alpha_{i-1}}{1+d_1/\alpha_1+\dots+d_{i-1}/\alpha_{i-1}}}.$$

Similarly, we can consider the iterated wreath product

$$(\dots(\mathbb{Z}^{d_0} \wr \mathbb{Z}^{d_1}) \wr \mathbb{Z}^{d_2}) \wr \dots) \wr \mathbb{Z}^{d_k},$$

starting with lamp group $\mathbb{Z}^{d_0}$ instead of $\mathbb{Z}_2$ and $q_0 = \mu_0$ in the domain of normal attraction of the rotationally symmetric α_0-stable distribution on $\mathbb{R}^{d_0}$. In this case, we obtain

$$\lim_{n\to\infty} [n^{\gamma_k/(1+\gamma_k)} \log^{1/(1+\gamma_k)} n]^{-1} \log q^{(n)}(e) = -c_k, \quad \gamma_k = d_1/\alpha_1 + \dots + d_k/\alpha_k.$$

The constant c_k can be obtained iteratively with $c_1 = k\left(\eta, \frac{d_0}{\alpha_0}\mathbf{1}_{\{y>0\}}\right)$ and $c_i = k\left(\upsilon_i, \widetilde{F}_i\right)$, with $\widetilde{F}_i$ as above for $2 < i \le k$.

4.3 Application to fastest decay under moment conditions

This section describes applications of Theorem 4.2 to the computation of the group invariants $\widetilde{\Phi}_{G,\rho}$ introduced in [1]. Recall that [14] introduce a group invariant Φ_G which is a decreasing function of n (defined up to the equivalence relation $\simeq$) such that

$$\phi^{(2n)}(e) \simeq \Phi_G(n)$$

for all finitely supported symmetric probability measure ϕ with generating support.

Let ρ be a function

$$\rho : G \to [1, \infty).$$

The weak ρ-moment of the probability measure μ is defined as

$$W(\rho, \mu) := \sup_{s>0} s\mu(x : \rho(x) > s).$$

Definition 4.6 (Definition 2.1 [1]: Fastest decay under weak ρ-moment)**.** Let G be a locally compact unimodular group. Fix a compact symmetric neighborhood Ω of e. Let $\widetilde{\mathcal{S}}^{\Omega,K}_{G,\rho}$ be the set of all symmetric continuous probability densities ϕ on G with the properties that $\|\phi\|_\infty \le K$ and $W(\rho, \phi d\lambda) \le K \sup_{\Omega^2}\{\rho\}$. Set

$$\widetilde{\Phi}^{\Omega,K}_{G,\rho}(n) := \inf\{\phi^{(2n)}(e) : \phi \in \widetilde{\mathcal{S}}^{\Omega,K}_{G,\rho}\}.$$

Here we will only consider the case when G is finitely generated and ρ is one of the power function $\rho_\alpha(x) = (1+|x|)^\alpha$, $\alpha \in (0, 2)$ where $|\cdot|$ is the word distance on a fixed Cayley graph of G. We are concerned with the decay of $\widetilde{\Phi}^{\Omega,K}_{G,\rho_\alpha}$ when n is large. By Proposition 1.2 [1], we can drop the reference to Ω and K. Lower bounds on $\widetilde{\Phi}_{G,\rho}$ follow from general comparison and subordination results, see [1]. Here, we are interested in obtaining upper bounds on $\widetilde{\Phi}_{G,\rho}$.

By definition, for any probability measure ϕ on G which satisfies the weak ρ-moment condition, $n \mapsto \phi^{(2n)}(e)$ provides an upper bound for $\widetilde{\Phi}_{G,\rho}$. When G is a wreath product $G = K \wr \mathbb{Z}^d$, we can use measures of the form $\phi = \nu * \mu * \nu$ and apply Theorem 4.2 to estimate $\phi^{(2n)}(e)$. Also because of the natural embedding of K and $\mathbb{Z}^d$ in the wreath product $K \wr \mathbb{Z}^d$, it's not hard to estimate the needed weak ρ-moment of ϕ. We shall see that, in certain cases, the measures ϕ of this type actually achieve the fastest decay rate given by $\widetilde{\Phi}_{G,\rho}$, up to the equivalence relation $\simeq$. This technique was already used in [1, Theorem 5.1] to determine $\widetilde{\Phi}_{\mathbb{Z}_2 \wr \mathbb{Z}^d, \rho_\alpha}$. In this case, the classical result of Donsker and Varadhan [7] is all one needs. In the examples below, we use Theorem 4.2 to obtain precise upper bounds on $\Phi_{K \wr \mathbb{Z}^d}$ in some other cases.

Example 4.5. In this example we consider $G = K \wr \mathbb{Z}^d$ when K is either finite or has polynomial growth or has exponential volume growth and $\Phi_K(n) \simeq \exp(-n^{1/3})$. The first case is already treated in [1]. We note that these three cases exhaust all possibilities when K is a polycyclic group. The third case also covers the situations when K is the Baumslag-Solitar group or the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$.

Theorem 4.7. *Fix $\alpha \in (0, 2)$. Let G be the group $K \wr \mathbb{Z}^d$.*

1. *Assume that K is finite. Then*

$$\log \widetilde{\Phi}_{G,\rho_\alpha}(n) \simeq -n^{d/(d+\alpha)}.$$

2. *Assume that K has polynomial volume growth. Then*

$$\log \widetilde{\Phi}_{G,\rho_\alpha}(n) \simeq -n^{d/(d+\alpha)}(\log n)^{\alpha/(d+\alpha)}.$$

3. *Assume that K has exponential growth and satisfies $\Phi_K(n) \simeq \exp(-n^{1/3})$. Then*

$$\log \widetilde{\Phi}_{G,\rho_\alpha}(n) \simeq -n^{(d+1)/(d+1+\alpha)}.$$

Proof. The lower bounds can be obtained by applying [1, Theorem 3.3]. For this purpose, one needs to compute the function Φ_G. For $q = \nu * \mu * \nu$ on $G = K \wr \mathbb{Z}^d$, where μ and ν are associated with simple random walk on $\mathbb{Z}^d$ and K respectively, we can apply Theorem 4.2 to obtain decay of $q^{(2n)}(e) \simeq \Phi_G(n)$. The case when K is finite is already treated in [13, 1]. When K is of polynomial volume growth, then $\nu^{(2n)}(o) \asymp n^{-\frac{D}{2}}$ and, as in Example 4.2,

$$\lim_{n\to\infty} \frac{1}{n^{\frac{d}{d+2}}(\log n)^{\frac{2}{d+2}}} \log q^{(2n)}(e) = -c_q.$$

If K is such that $\Phi_K(n) \simeq \exp(-n^{1/3})$ then Example 4.1 yields

$$\log q^{(2n)}(e) \simeq \log \Phi_G(n) \simeq -n^{\frac{d+1}{d+3}}.$$

These estimates on Φ_G allow us to appeal to [1, Theorem 3.3] to obtain the stated lower bounds for Φ_{G,ρ_α}.

To prove the stated upper bounds, it suffices to exhibit a probability measure measure in $\widetilde{\mathcal{S}}_{G,\rho_\alpha}$ that has the proper decay. Take

$$\mu_\alpha(x) = \frac{c_\alpha}{(1+|x|)^{\alpha+1}}$$

to be the symmetric α-stable like probability measure on $\mathbb{Z}^d$. Then μ_α is in the domain of normal attraction of the rotationally symmetric α-stable distribution on $\mathbb{R}^d$ and it has a finite weak α-moment.

In the case when K is of polynomial volume growth, take $q_\alpha = \nu * \mu_\alpha * \nu$, where ν is simple random walk on K. Then $\nu * \mu_\alpha * \nu$ has weak α-moment and, by Example 4.2,

$$\lim_{n\to\infty} \frac{1}{n^{\frac{d}{d+\alpha}}(\log n)^{\frac{\alpha}{d+\alpha}}} \log q_\alpha^{(2n)}(e) = -c_{q_\alpha}.$$

Therefore in this case

$$\log \widetilde{\Phi}_{G,\rho_\alpha}(n) \leq -cn^{\frac{d}{d+\alpha}}(\log n)^{\frac{\alpha}{d+\alpha}}.$$

This matches the previously proved lower bound.

In the second case, when K has exponential growth, let U be a symmetric generating set of K. As in [1, Theorem 4.10], pick $p_i = c_\alpha 4^{-i\alpha}$ with $\sum_1^\infty p_i = 1$ and set

$$\nu_\alpha = \sum_{i=1}^{\infty} \frac{p_i}{|U^{4^i}|} \mathbb{1}_{U^{4^i}}.$$

Then ν_α has weak α-moment on K and, by [1, Theorem 4.1]

$$\nu_\alpha^{(n)}(e_K) \le \exp(-cn^{\frac{1}{1+\alpha}}).$$

Then $\nu_\alpha * \mu_\alpha * \nu_\alpha$ has weak α-moment on G. Applying Theorem 4.2 and the computations of Example 4.1 to $q_\alpha = \nu_\alpha * \mu_\alpha * \nu_\alpha$, we obtain

$$\lim_{n\to\infty} \sup \frac{1}{n^{\frac{d+1}{d+\alpha+1}}} \log q_\alpha^{(2n)}(e) \le -c_{q_\alpha}.$$

This gives the desired upper bound on Φ_{G,ρ_α}. $\square$

Example 4.6. Consider the iterated wreath product $G = (\dots(K\wr\mathbb{Z}^{d_1})\wr\dots)\wr\mathbb{Z}^{d_r}$, with $d_i \in \mathbb{N}_+$. Fix $\alpha \in (0,2)$. If K is finite then we we have

$$\log \widetilde{\Phi}_{G,\rho_\alpha}(n) \simeq -n^{\frac{d_1+\dots+d_r}{\alpha+d_1+\dots+d_r}}.$$

If K has polynomial volume growth, then

$$\log \widetilde{\Phi}_{G,\rho_\alpha}(n) \simeq -cn^{\frac{d_1+\dots+d_r}{\alpha+d_1+\dots+d_r}}(\log n)^{\frac{\alpha}{\alpha+d_1+\dots+d_r}}.$$

These are the results stated as Theorem 1.4 in the introduction.

Proof. As in the previous example, the lower bounds follows from [1, Theorem 3.3] and a lower bound on $\log\Phi_G$. By example 4.4,

$$\log \Phi_G(n) \simeq n^{\frac{d_1+\dots+d_r}{2+d_1+\dots+d_r}}(\log n)^{\frac{2}{2+d_1+\dots+d_r}}.$$

Hence [1, Theorem 3.3] gives

$$\log \widetilde{\Phi}_{G,\rho_\alpha}(n) \ge -C_\alpha n^{\frac{d_1+\dots+d_r}{\alpha+d_1+\dots+d_r}}(\log n)^{\frac{\alpha}{\alpha+d_1+\dots+d_r}}.$$

For the upper bound, let $\mu_{\alpha,i}$ be a symmetric α-stable like probability measure on $\mathbb{Z}^{d_i}$. Let $q_{\alpha,1} = \mu_{\alpha,0} * \mu_{\alpha,1} * \mu_{\alpha,0}$, and iteratively define $q_{\alpha,i+1} = q_{\alpha,i} * \mu_{\alpha,i+1} * q_{\alpha,i}$. Then it's clear that $q_{\alpha,r}$ has a finite weak α-moment on G and, as in example 4.4,

$$\lim_{n\to\infty} \frac{1}{n^{\frac{d_1+\dots+d_r}{\alpha+d_1+\dots+d_r}}(\log n)^{\frac{\alpha}{\alpha+d_1+\dots+d_r}}} \log q_{\alpha,r}^{(2n)} = -c_{\alpha,r}.$$

Therefore

$$\log \widetilde{\Phi}_{G,\rho_\alpha}(n) \le -cn^{\frac{d_1+\dots+d_r}{\alpha+d_1+\dots+d_r}}(\log n)^{\frac{\alpha}{\alpha+d_1+\dots+d_r}}.$$

$\square$

5 Donsker and Varadhan type large deviations

The goal of this section is to outline the proof of Theorem 3.9, the key result of this article. The proof follows [7] closely. Several other classical sources are also needed to put together the necessary details.

5.1 Statement of the large deviation principle in L^1

On $\mathbb{Z}^d$, we fix a symmetric probability measure μ and an operator-stable law η such that the convergence assumption (C-B_n) is satisfied.

We need to introduce some notation from [7] in order to state the results. Let π be the projection map $\pi : \mathbb{R}^d \to \mathbb{R}^d/\mathbb{Z}^d$, and let $\mathbb{T}$ denote the d-dimensional torus which we also identify with the fundamental domain $[-\frac{1}{2}, \frac{1}{2})^d$.

For $\lambda > 0$, set

$$\mathcal{L}_\lambda^{(n)} = \pi\left(B_{\lfloor \lambda a_n \rfloor}^{-1}(\mathbb{Z}^d)\right).$$

That is, we take the image of the original lattice $\mathbb{Z}^d$ under the transformation $B_{\lfloor \lambda a_n \rfloor}^{-1}$, and project it to the torus $\mathbb{T}$. Then $\mathcal{L}_\lambda^{(n)}$ is a cocompact lattice on $\mathbb{T}$ and the volume of the fundamental domain $\mathbb{T}/\mathcal{L}_\lambda^{(n)}$ is $\left|\det B_{\lfloor \lambda a_n \rfloor}^{-1}\right|$. This is the case because we assume that the matrices B_m, $m = 1, 2\ldots$, have integer entries so that $B_m \mathbb{Z}^d \subset \mathbb{Z}^d$.

In what follows, symbols decorated with$^{\sim}$ are always used to describe quantities associated with the projected random walk on the torus. Note that the construction depends on the choice of sequence a_n and parameter λ, for simplicity we will drop reference to a_n and λ when no confusion arises.

Under the projection map π, we can push forward the measure $B_{\lfloor \lambda a_n \rfloor}^{-1}\mu$ on $B_{\lfloor \lambda a_n \rfloor}^{-1}(\mathbb{Z}^d)$ to a measure $\widetilde{\mu}_{n,\lambda}$ on $\mathcal{L}_\lambda^{(n)}$, that is

$$\widetilde{\mu}_{n,\lambda}(y) = \sum_{x \in \mathbb{Z}^d : \pi\left(B_{\lfloor \lambda a_n \rfloor}^{-1}(x)\right) = y} \mu(x).$$

Let $\widetilde{S}_k^{(n)}$ be the random walk on $\mathcal{L}_\lambda^{(n)}$ associated with $\widetilde{\mu}_{n,\lambda}$, starting at 0. It's easy to check that

$$\widetilde{S}_k^{(n)} \overset{\text{law}}{=} \pi\left(B_{\lfloor \lambda a_n \rfloor}^{-1}(S_k)\right).$$

Consider the occupation time measure $\widetilde{L}_k^{(n)}$ defined as

$$\widetilde{L}_k^{(n)}(A) = \frac{1}{k}\sum_{j=1}^{k} \chi_A\left(\widetilde{S}_k^{(n)}\right),$$

for any Borel set A in $\mathbb{T}$.

Let $\mathcal{M}_1(\mathbb{T})$ be the space of probability measures on $\mathbb{T}$ endowed with the weak topology. Let $P_k^{(n)}$ be the distribution of $\widetilde{L}_k^{(n)}$ in $\mathcal{M}_1(\mathbb{T})$, a measure on measures. Define the scaled indicator function $\chi_n : [-\frac{1}{2}, \frac{1}{2})^d \to \mathbb{R}$ by setting

$$\chi_n(x) = |\det B_{\lfloor \lambda a_n \rfloor}|^{-1} \chi_{B_{\lfloor \lambda a_n \rfloor}^{-1}([-\frac{1}{2}, \frac{1}{2})^d)}.$$

Define

$$\widetilde{L}_k^n = \widetilde{L}_k^{(n)} * \chi_n.$$

Let $\widetilde{P}_k^n$ be the distribution of $\widetilde{L}_k^n$ in $\mathcal{M}_1(\mathbb{T})$. With this mollification, $\widetilde{L}_k^n$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{T}$. Let $\widetilde{f}_{k,\lambda}^{(n)}$ denote the density of $\widetilde{P}_k^n$ with respect to Lebesgue measure. Let $Q_{k,\lambda}^{(n)}$ be the distribution of $\widetilde{f}_{k,\lambda}^{(n)}$ in $L_1(\mathbb{T})$.

Theorem 5.1 (Large deviation principle in $L^1(\mathbb{T})$)**.** *Assume that the convergence assumption* (C-B_n) *is satisfied. Let a_n to be any sequence of positive integers increasing to infinity and satisfying $a_n |\det B_{a_n}| \le n$. Let $Q_{n,\lambda}^{(n)}$ be the distribution of $\widetilde{f}_{n,\lambda}^{(n)}$ on $L_1(\mathbb{T})$. Then we have the large deviation principle in the strong $L_1(\mathbb{T})$ topology. Namely, for any Borel set D in $L_1(\mathbb{T})$,*

$$\begin{aligned}
-\lambda^{-1} \inf_{f\in D^\circ} I_{L_{\widetilde{\eta}}}(f) &\le \liminf_{n\to\infty} \frac{1}{n/a_n} \log Q_{n,\lambda}^{(n)}(D) \\
&\le \limsup_{n\to\infty} \frac{1}{n/a_n} \log Q_{n,\lambda}^{(n)}(D) \le -\lambda^{-1} \inf_{f\in \overline{D}} I_{L_{\widetilde{\eta}}}(f),
\end{aligned}$$

and the rate function is given by

$$I_{L_{\widetilde{\eta}}}(f) = -\inf_{u\in\mathcal{U}_{\mathbb{T}}} \int_{\mathbb{T}} \frac{L_{\widetilde{\eta}}u}{u}(x) f(x)dx = \mathcal{E}_{\widetilde{\eta}}(\sqrt{f}, \sqrt{f}).$$

This result will be useful in the upper bound direction. To obtain a lower bound, we need to have a version with Dirichlet boundary condition.

Let $L_k^{(n)}$ be the occupation time measure of the random walk $S_k^{(n)} = B_{a_n}^{-1}(S_k)$. Perform the same mollification as above but on $\mathbb{R}^d$, setting

$$L_k^n = L_k^{(n)} * \chi_n.$$

Then L_k^n is absolutely continuous with respect to Lebesgue measure. Let $f_k^{(n)}$ denotes the corresponding density. Let Ω be a bounded domain in $\mathbb{R}^d$ such that $0\in\Omega$ and $\partial\Omega$ has Lebesgue measure 0. For any Borel set $A\subset L_1(\Omega)$, define

$$Q_{k,\Omega}^{(n)}(A) := P\left(f_k^{(n)} \in A\right).$$

That is, $Q_{k,\Omega}^{(n)}$ is the distribution of the occupation time measure of $S_j^{(n)}$ at time k with Dirichlet boundary on $\partial\Omega$. As in the case of the projected version, we have the following large deviation principle.

Theorem 5.2 (Large deviation principle in $L^1(\Omega)$)**.** *Under the convergence assumption* (C-B_n)*, let a_n be any sequence of positive integers increasing to infinity satisfying $a_n |\det B_{a_n}| \le n$. Let $Q_{n,\Omega}^{(n)}$ be the distribution of $f_n^{(n)}$ in $L_1(\Omega)$. Then we have large deviation principle in the strong $L_1(\Omega)$ topology. Namely, for any Borel set $A\subset L_1(\Omega)$,*

$$\begin{aligned}
-\inf_{f\in A^\circ} I_{L_\eta}(f) &\le \liminf_{n\to\infty} \frac{1}{n/a_n} \log Q_{n,\Omega}^{(n)}(A) \\
&\le \limsup_{n\to\infty} \frac{1}{n/a_n} \log Q_{n,\Omega}^{(n)}(A) \le -\inf_{f\in\overline{A}} I_{L_\eta}(\eta),
\end{aligned}$$

and

$$I_{L_\eta}(f) = -\inf_{u\in\mathcal{U}_\Omega}\int_\Omega \frac{L_\eta u}{u}(x)f(x)dx = \mathcal{E}_\eta(\sqrt{f},\sqrt{f}).$$

The outlined of the proof of these results is given in Section 5.3.

5.2 Asymptotics of functional expressions

Fix a symmetric probability measure μ on $\mathbb{Z}^d$ and an operator-stable law η on $\mathbb{R}^d$ such that the convergence assumption (C-B_n) and scaling assumption (S-B_n-a_n) of Definitions 3.1-3.6 are satisfied. In particular, in what follows, (a_n) is the non-decreasing and regularly varying sequence of integers provided by Definition 3.6 (see also Proposition 3.7). The functions F and $\widetilde{F}$ are as in Definition 3.6. Let $(l(n,x))_{x\in\mathbb{Z}^d}$ be the occupation time vector up to time n for the random walk driven μ.

Proposition 5.3. *Under the above hypotheses, we have the lower bound*

$$\begin{aligned}\liminf_{n\to\infty}\frac{a_n}{n}\log E\left(\exp\left(-\sum_{x\in\mathbb{Z}^d}F(l(n,x))\right)\mathbf{1}_{B_{a_n}\Omega}\right)\\ \geq\ -\inf_{f\in\mathcal{F}_\Omega}\left\{\mathcal{E}_\eta(\sqrt{f},\sqrt{f})+\int_\Omega\widetilde{F}(f(x))dx\right\}.\end{aligned}$$

Proof. The proof is essentially the same as for [2, Lemma 4.2]. Use the lower bound in 5.2 and Varadhan's lemma. See [17, Theorems 2.2, 2.3]. □

Proposition 5.4. *Under the convergence assumption (C-B_n) and the scaling assumption (S-B_n-a_n), we have upper bound*

$$\begin{aligned}&\limsup_{n\to\infty}\frac{a_n}{n}\log E\left(\exp\left(-\sum_{x\in\mathbb{Z}^d}F(l(n,x))\right)\right)\\ \leq\ &-\sup_{\lambda>0}\inf_{f\in\mathcal{F}_\mathbb{T}}\left\{\lambda^{-1}\mathcal{E}_{\widetilde{\eta}}(\sqrt{f},\sqrt{f})+c_0\lambda^{(1-\gamma)\operatorname{tr}E}\int_\mathbb{T}\widetilde{F}(f(x))dx\right\}.\end{aligned}$$

Proof. First, since F is sub-additive, write

$$\begin{aligned}E\left(\exp\left(-\sum_{x\in\mathbb{Z}^d}F(l(n,x))\right)\right) &\leq E\left(\exp\left(-\sum_{y\in\mathcal{L}_\lambda^{(n)}}F\left(\widetilde{l}(n,y)\right)\right)\right)\\ &= E_{Q_{n,\lambda}^{(n)}}\left(\exp\left(-\det(B_{\lambda a_n})\int_\mathbb{T}F\left(\frac{n}{\det(B_{\lambda a_n})}f(x)\right)dx\right)\right).\end{aligned}$$

Next, we follow the line of reasoning used to prove the Corollary of Theorem 6 in [7], using the large deviation upper bound in $L_1(\mathbb{T})$ and Varadhan's lemma.

From the (lower bound part of) the scaling assumption and regular variation property of $\det B_{a_n}$, we have for any parameter $\lambda > 0$,

$$\liminf_{n\to\infty} \frac{a_n \det(B_{\lambda a_n})}{n} F\left(\frac{n}{\det(B_{\lambda a_n})} y\right) \geq \lambda^{(1-\gamma)\operatorname{tr} E} \widetilde{F}(y), \; y > 0.$$

Setting $\mathbf{D}_n = \det(B_{\lambda a_n})$, we obtain

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{a_n}{n} \log E\left(\exp\left(-\sum_{x\in\mathbb{Z}^d} F(l(n,x))\right)\right) \\
\leq\; & \limsup_{n\to\infty} \frac{a_n}{n} \log E_{Q^{(n)}_{n,\lambda}}\left(\exp\left(-\mathbf{D}_n \int_{\mathbb{T}} F\left(\frac{n}{\mathbf{D}_n} f(x)\right) dx\right)\right) \\
=\; & \limsup_{n\to\infty} \frac{a_n}{n} \log E_{Q^{(n)}_{n,\lambda}}\left(\exp\left(-\frac{n}{a_n} \int_{\mathbb{T}} \frac{a_n \mathbf{D}_n}{n} F\left(\frac{n}{\mathbf{D}_n} f(x)\right) dx\right)\right) \\
\leq\; & -\inf_{f\in\mathcal{F}_{\mathbb{T}}} \left\{\lambda^{-1}\mathcal{E}_{\widetilde{\eta}}(\sqrt{f},\sqrt{f}) + \lambda^{(1-\gamma)\operatorname{tr} E} \int_{\mathbb{T}} \widetilde{F}(f(x))dx\right\}.
\end{aligned}$$

The last step comes from Varadhan's lemma. Since the choice of parameter λ is arbitrary, we can optimize over all $\lambda > 0$. $\square$

The following lemma is proved in the appendix. It shows that the constants appearing in the upper and lower bounds actually match up. In particular, since this constant appears as both a sup and an inf of some nonnegative quantities, it follows clearly that the constant $k(\eta, \widetilde{F})$ defined below takes value in $(0,\infty)$.

Lemma 5.5. *Suppose $\widetilde{F}$ is a homogeneous function with exponent $\gamma \in [0,1]$, that is $\widetilde{F}(0) = 0, \widetilde{F}(y) = \widetilde{F}(1)y^\gamma$ for $y > 0$; and η is a full operator-stable law with exponent E. Then there exists a constant $k(\eta, \widetilde{F} \in (0,\infty)$ such that*

$$\begin{aligned}
k(\eta,\widetilde{F}) &= \sup_{\lambda>0} \inf_{f\in\mathcal{F}_{\mathbb{T}}} \left\{\lambda^{-1}\mathcal{E}_{\widetilde{\eta}}(\sqrt{f},\sqrt{f}) + \lambda^{(1-\gamma)\operatorname{tr} E} \int_{\mathbb{T}} \widetilde{F}(f(x))dx\right\} \\
&= \inf_{\Omega\in\mathcal{G}} \inf_{f\in\mathcal{F}_\Omega} \left\{\mathcal{E}_\eta(\sqrt{f},\sqrt{f}) + \int_\Omega \widetilde{F}(f(x))dx\right\}.
\end{aligned}$$

5.3 Proof of the large deviation principle in L^1

In this section we indicate how to adapt [7] to prove the large deviation principles as stated in Theorems 5.1 and 5.2. Throughout this section we assume

$$B_n^{-1}\mu^{(n)} \Longrightarrow \eta. \tag{C-B_n}$$

First we establish a large deviation principle for $\widetilde{P}^{(n)}_{n,\lambda}$ and $P^{(n)}_{n,\Omega}$ in the weak topology.

Proposition 5.6 (Similar to [7, Theorem 3]). *Let C be a closed of $\mathcal{M}_1(\mathbb{T})$. Then*

$$\limsup_{n\to\infty} \frac{1}{n/a_n} \log \widetilde{P}^{(n)}_{n,\lambda}(C) \le -\lambda^{-1} \inf_{\nu\in C} I_{L_{\widetilde{\eta}}}(\nu),$$

where

$$I_{L_{\widetilde{\eta}}}(\nu) = -\inf_{u\in\mathcal{U}_{\mathbb{T}}} \int_{\mathbb{T}} \frac{L_{\widetilde{\eta}}u}{u}(x)d\nu(x).$$

Similarly, if C is compact in $\mathcal{M}_1(\Omega)$,

$$\limsup_{n\to\infty} \frac{1}{n/a_n} \log P^{(n)}_{n,\Omega}(C) \le -\inf_{\nu\in C} I_{L_\eta}(\nu),$$

where

$$I_{L_\eta}(\nu) = -\inf_{u\in\mathcal{U}_\Omega} \int_\Omega \frac{L_\eta u}{u}(x)d\nu(x).$$

We have the following Feynman-Kac estimates as consequences of functional limit theorem. For the proof, adapt the arguments given in [3].

Theorem 5.7 (Similar to [3, Theorem 7.1.1]). *For any bounded, continuous function f on $\mathbb{R}^d$, and for any sequence (a_n) satisfying $a_n \to \infty$ and $a_n = o(n)$ as $n \to \infty$, we have*

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{n/a_n} \log E\left(\exp\left(\frac{1}{a_n}\sum_{k=1}^n f(B_{a_n}^{-1}S_k)\right)\right)&\\ \ge \sup_{g\in\mathcal{F}}\left\{\int_{\mathbb{R}^d} f(x)g(x)dx - \mathcal{E}_\eta(\sqrt{g},\sqrt{g})\right\}&.\end{aligned}$$

Corollary 5.8. *For any $f \in C(\mathbb{T})$,*

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{n/a_n} \log E\left(\exp\left(\frac{1}{a_n}\sum_{k=1}^n f(\widetilde{S}^{(n)}_k)\right)\right)&\\ \ge \sup_{g\in\mathcal{F}_{\mathbb{T}}}\left\{\int_{\mathbb{T}} f(x)g(x)dx - \lambda^{-1}\mathcal{E}_{\widetilde{\eta}}(\sqrt{g},\sqrt{g})\right\}&.\end{aligned}$$

Similarly, for $f \in C(\Omega)$,

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{n/a_n} \log E\left(\exp\left(\frac{1}{a_n}\sum_{k=1}^n f(S^{(n)}_k)\right)\right)&\\ \ge \sup_{g\in\mathcal{F}_\Omega}\left\{\int_\Omega f(x)g(x)dx - \mathcal{E}_\eta(\sqrt{g},\sqrt{g})\right\}&.\end{aligned}$$

Apply Varadhan's lemma ([17, Theorem 2.2]) to the upper bound in Proposition 5.6. Together with the lower bound, this yields the asymptotics for the log-moment generating functions stated in the following Proposition. Compare with [9, Lemma 6.1] which treats simple random walk on $\mathbb{Z}^d$.

Proposition 5.9. *For the projected occupation measure, for any $f \in C(\mathbb{T})$,*

$$\lim_{n\to\infty} \frac{1}{n/a_n} \log E\left(\exp\left(\frac{n}{a_n} < f, \widetilde{L}_n^{(n)} >\right)\right)$$
$$= \sup_{g\in\mathcal{F}_{\mathbb{T}}} \left\{ \int_{\mathbb{T}} f(x)g(x)dx - \lambda^{-1}\mathcal{E}_{\widetilde{\eta}}(\sqrt{g}, \sqrt{g}) \right\}.$$

For the occupation measure with Dirichlet boundary condition, for any function $f \in C(\Omega)$,

$$\lim_{n\to\infty} \frac{1}{n/a_n} \log E\left(\exp\left(\frac{n}{a_n} < f, L_n^{(n)} >\right)\right)$$
$$= \sup_{g\in\mathcal{F}_{\Omega}} \left\{ \int_{\Omega} f(x)g(x)dx - \mathcal{E}_{\eta}(\sqrt{g}, \sqrt{g}) \right\}.$$

By the Gartner-Ellis theorem (e.g., [4, Theorem 4.5.20]), we obtain the large deviation principle in the weak topology stated in the following Theorem. Compare with [9, Lemma 3.1] which treats the case of simple random walk on $\mathbb{Z}^d$.

Theorem 5.10. *For any Borel set B in $\mathcal{M}_1(\mathbb{T})$,*

$$\begin{aligned} -\lambda^{-1} \inf_{f\in B^\circ} I_{L_{\widetilde{\eta}}}(f) &\leq \liminf_{n\to\infty} \frac{1}{n/a_n} \log \widetilde{P}_{n,\lambda}^{(n)}(B) \\ &\leq \limsup_{n\to\infty} \frac{1}{n/a_n} \log \widetilde{P}_{n,\lambda}^{(n)}(B) \leq -\lambda^{-1} \inf_{f\in\overline{B}} I_{L_{\widetilde{\eta}}}(f). \end{aligned}$$

Similarly, for any Borel set A in $\mathcal{M}_1(\Omega)$,

$$\begin{aligned} -\inf_{f\in A^\circ} I_{L_{\eta}}(f) &\leq \liminf_{n\to\infty} \frac{1}{n/a_n} \log P_{n,\Omega}^{(n)}(A) \\ &\leq \limsup_{n\to\infty} \frac{1}{n/a_n} \log P_{n,\Omega}^{(n)}(A) \leq -\inf_{f\in\overline{A}} I_{L_{\eta}}(\eta). \end{aligned}$$

Next, following [7], we use the local limit theorem to upgrade the large deviation principle in the weak topology to a result in the strong L^1-topology. This is a rather technical task. As shown in [7, Theorem 6], the key point is to obtain a super-exponential estimate on the L^1-distance of the density function to its smooth mollification. This in turn needs uniform properties of transition probabilities as provided by the local limit theorem.

Let $\{\psi_\epsilon\}$, $\epsilon \to 0$, be an approximation of the identity on $\mathbb{R}^d$ with ψ_ϵ is smooth, symmetric, compactly supported inside $(-\frac{\epsilon}{2}, \frac{\epsilon}{2})^d$. Thinking of ψ_ϵ also as a function on $\mathbb{T}$, set $K_\epsilon : L^1(\mathbb{T}) \to L^1(\mathbb{T})$ as

$$K_\epsilon f(x) = \int_{\mathbb{T}} f(y)\psi_\epsilon(x-y)dy.$$

Theorem 5.11 (Similar to [7, Theorem 5]). *For every $\delta > 0$, $\lambda > 0$,*

$$\limsup_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n/a_n} \log Q^{(n)}_{n,\lambda} \left(f : \int_{\mathbb{T}} |K_\epsilon f(x) - f(x)|\, dx \geq \delta \right) = -\infty.$$

Now we adapt to our situation the sequence of lemmas in [7] that are used to prove this theorem.

The first lemma is an elementary way to select a δ-net of functions.

Lemma 5.12 (Similar to [7, Lemma 4.1]). *Let $M_{n,\epsilon} \subset C(\mathbb{T})$ be the set of functions*

$$M_{n,\epsilon} = \{V = (K_\epsilon - I)\chi_n g : g \in C(\mathbb{T}), \|g\|_\infty \leq 1\}.$$

For any $\delta > 0$, there exist functions $V_1, ..., V_J$ such that for any $V \in M_{n,\epsilon}$,

$$\inf_{1\leq i\leq J} \sup_{x\in\mathcal{L}^{(n)}} |V(x) - V_i(x)| \leq \frac{\delta}{2},$$

and

$$J = J(n, \epsilon, \delta) \leq \left(\frac{8}{\delta} + 1\right)^{|\det B_{a_n}|}.$$

The second lemma provides a uniform control of the transition probabilities. Such uniform control appears as Assumption (U) in [5, Section 4.1] and [4, Section 6.3] to obtain the large deviation principle in L^1.

Lemma 5.13 (Similar to [7, Lemma 4.2]). *There exists $n_0 \in \mathbb{N}$ and constant $c < \infty$ such that for all $n \geq n_0$,*

$$\sup_{x\in\mathcal{L}^{(n)}} \widetilde{\mu}_n^{*a_n}(x) \leq c \inf_{x\in\mathcal{L}^{(n)}} \widetilde{\mu}_n^{*a_n}(x).$$

Proof. Recall the local limit theorem in [10],

$$\lim_{n\to\infty} \sup_{x\in\mathbb{Z}^d} |\det B_n| \cdot \left|\mu^{(n)}(x) - \left|\det B_n^{-1}\right| g(B_n^{-1}x)\right| = 0.$$

Applying this local limit result along the sequence $\{\lfloor \lambda a_n \rfloor\}$ and projecting onto $\mathbb{T}$, we have

$$\limsup_{n\to\infty} \sup_{y\in\mathcal{L}^{(n)}} ||\det B_{a_n}| \widetilde{\mu}_n^{*a_n}(y) - \widetilde{g}_{\lambda^{-1}}(y)| = 0.$$

Since the density g is continuous, the desired result follows. ◻

As a consequence of Lemma 5.13, we have the following uniform estimate with respect to the starting point.

Lemma 5.14 (Similar to [7, Lemma 4.3])**.** *Let x and y be any two points in $\mathcal{L}_\lambda^{(n)}$. There exists an integer n_0 such that for any $n \geq n_0$ and any $\theta > 0$,*

$$E_y\left[\exp\left(\theta\sum_{k=1}^{n} V(\widetilde{S}_k^{(n)})\right)\right] \leq C_{n,\theta} E_x\left[\exp\left(\theta\sum_{k=1}^{n} V(\widetilde{S}_k^{(n)})\right)\right],$$

where $C_{n,\theta} = c\exp(4\theta a_n)$ and c and n_0 are as in Lemma 4.2.

Lemma 5.15 (Similar to [7, Lemma 4.4])**.** *There exists an integer n_0 such that for any $n \geq n_0$ and any $\theta > 0$,*

$$E_0\left[\exp\left(\theta\sum_{k=1}^{n} V(\widetilde{S}_k^{(n)})\right)\right] \leq C_{n,\theta}\exp\left(n\widetilde{I}_n^*(\theta V)\right),$$

where $C_{n,\theta}$ is as in Lemma 5.14 *and $\widetilde{I}_n^*$ is Legendre transform of*

$$\widetilde{I}_n(\eta) = -\inf_{u\in\mathcal{U}_{\mathbb{T}}}\int_{\mathbb{T}} \log\frac{\widetilde{\mu}_n u}{u} d\eta,$$

that is,

$$\widetilde{I}_n^*(\theta V) = \sup_{\eta\in\mathcal{M}_1(\mathbb{T})}\left\{\int_{\mathbb{T}} \theta V d\eta - \widetilde{I}_n(\eta)\right\}.$$

Finally, we need the following technical lemma that controls error terms as $n \to \infty$:

Lemma 5.16 (Similar to [7, Lemma 4.5])**.** *Let η be a probability measure on $\mathcal{L}^{(n)}$ such that $\widetilde{I}_n(\eta) \leq \frac{\sigma}{a_n}$, where $\sigma > 0$. Let $V = (K_\epsilon - I)\chi_n g$ where $g \in C(\mathbb{T}), \|g\|_\infty \leq B$. Then for any $t > 0$,*

$$\int_{\mathbb{T}} V d\eta \leq B[2h(t\sigma) + 2\Delta_t(n) + k_t(\epsilon)],$$

where

$$h(l) := 2\inf_{a>0}\frac{l + a - \log(1+a)}{a}$$

and

$$\Delta_t(n) = \int_{\mathbb{T}}\left|\chi_n * \widetilde{\mu}_n^{(\lfloor ta_n\rfloor)}(x) - \widetilde{g}_{t/\lambda}(x)\right| dx,$$

$$k_t(\epsilon) = \sup_{y\in(-\frac{\epsilon}{2},\frac{\epsilon}{2})^d}\int_{\mathbb{T}}\left|\widetilde{g}_{t/\lambda}(x-y) - \widetilde{g}_{t/\lambda}(x)\right| dx.$$

Moreover, we have that $h(l) \to 0$ as $l \to 0$ and, for any fixed $t > 0$, $\Delta_t(n) \to 0$ as $n \to \infty$ and $k_t(\epsilon) \to 0$ as $\epsilon \to 0$.

With these five lemmas, the line of reasoning used in [7, Theorem 5] gives us Theorem 5.11.

6 Appendix

6.1 Proof of regular variation properties

In this subsection we deduce from regular variation property of $\{B_n\}$ the technical proposition 3.7. The idea closely follows [2].

Proof of Proposition 3.7. By [11, Theorem 1.10.19], under the convergence assumption

$$B_n^{-1}\mu^{(n)} \Longrightarrow \eta,$$

there exists a modified normalization sequence $\{B'_n = B_n S_n\}$ perturbed by s suitable $S_n \in \mathrm{Ivn}(\eta)$ such that

$$(B'_n)^{-1}\mu^{(n)} \Longrightarrow \eta$$

(this is clear since $S_n \in \mathrm{Ivn}(\eta)$) and $\{B'_n\}$ possesses regular variation property

$$B'_n(B'_{\lfloor nt \rfloor})^{-1} \to t^{-E}$$

where the convergence is uniform in t on compact subsets of $\mathbb{R}_+^\times$. Since $S_n \in \mathrm{Ivn}(\eta)$ and $\mathrm{Inv}(\eta)$ is a compact group, we must have $\det S_n = 1$, $\det B'_n = \det B_n$. Hence we can replace B_n in the scaling assumption by B'_n and we have

$$\lim_{n\to\infty} \frac{a_n \det(B'_{a_n})}{n} F\left(\frac{n}{\det(B'_{a_n})} y\right) = \widetilde{F}(y),$$

uniformly over compact sets in $(0, \infty)$.

Set

$$\widetilde{F}_n(y) := \frac{a_n \det(B'_{a_n})}{n} F\left(\frac{n}{\det(B'_{a_n})} y\right).$$

The scaling assumption now reads $\lim_{n\to\infty} \widetilde{F}_n(y) = \widetilde{F}(y)$. Note that by assumption, $F : [0, \infty) \to [0, \infty)$ is a concave, increasing function with $F(0) = 0$, therefore both $\widetilde{F}_n$ and $\widetilde{F}$ are concave, non-decreasing, not identically zero with value 0 at 0. Hence $\widetilde{F}_n$ and $\widetilde{F}$ are continuous and strictly positive in $(0, \infty)$, and by concavity, $y \to \frac{\widetilde{F}_n(y)}{y}$ and $y \to \frac{\widetilde{F}(y)}{y}$ are both non-increasing functions.

Now we'll show that for any $\lambda \in (0, 1)$, $\frac{a_{\lfloor \lambda n \rfloor}}{a_n}$ tends to a finite non-zero limit as $n \to \infty$. Fix a $y > 0$ and write

$$\begin{aligned}
\widetilde{F}_{\lfloor \lambda n \rfloor}(y) &= \frac{a_{\lfloor \lambda n \rfloor} \det(B'_{a_{\lfloor \lambda n \rfloor}})}{\lfloor \lambda n \rfloor} F\left(\frac{\lfloor \lambda n \rfloor}{\det(B'_{a_{\lfloor \lambda n \rfloor}})} y\right) \\
&= \frac{a_{\lfloor \lambda n \rfloor} \det(B'_{a_{\lfloor \lambda n \rfloor}})}{\lfloor \lambda n \rfloor} F\left(\frac{n}{\det(B'_{a_n})} \frac{\lfloor \lambda n \rfloor / n}{\det(B'_{a_{\lfloor \lambda n \rfloor}}) / \det(B'_{a_n})} y\right) \\
&= \frac{a_{\lfloor \lambda n \rfloor}}{a_n} \frac{\det(B'_{a_{\lfloor \lambda n \rfloor}}) / \det(B'_{a_n})}{\lfloor \lambda n \rfloor / n} \widetilde{F}_n\left(\frac{\lfloor \lambda n \rfloor / n}{\det(B'_{a_{\lfloor \lambda n \rfloor}}) / \det(B'_{a_n})} y\right).
\end{aligned}$$

As $\{a_n\}$ is an increasing sequence and $\det(B_n)$ is non-decreasing with respect to n, we have $\det(B'_{a_{\lfloor \lambda n \rfloor}})/\det(B'_{a_n}) \leq 1$ Since $y \to \frac{\widetilde{F}_n(y)}{y}$ is non-increasing, we have

$$\frac{\widetilde{F}_n\left(\frac{\lfloor \lambda n \rfloor / n}{\det(B'_{a_{\lfloor \lambda n \rfloor}})/\det(B'_{a_n})} y\right)}{\frac{\lfloor \lambda n \rfloor / n}{\det(B'_{a_{\lfloor \lambda n \rfloor}})/\det(B'_{a_n})} y} \leq \frac{\widetilde{F}_n(\frac{\lfloor \lambda n \rfloor}{n} y)}{\frac{\lfloor \lambda n \rfloor}{n} y}.$$

Therefore

$$\widetilde{F}_{\lfloor \lambda n \rfloor}(y) \leq \frac{a_{\lfloor \lambda n \rfloor}}{a_n} \frac{\widetilde{F}_n\left(\frac{\lfloor \lambda n \rfloor}{n} y\right)}{\frac{\lfloor \lambda n \rfloor}{n}}.$$

Letting $n \to \infty$ on both sides, the scaling assumption yields

$$\widetilde{F}(y) \leq \lim \inf_{n \to \infty} \frac{a_{\lfloor \lambda n \rfloor}}{a_n} \frac{1}{\lambda} \widetilde{F}(\lambda y).$$

Therefore

$$\lim \inf_{n \to \infty} \frac{a_{\lfloor \lambda n \rfloor}}{a_n} \geq \frac{\lambda \widetilde{F}(y)}{\widetilde{F}(\lambda y)}.$$

From the assumption that $\{a_n\}$ is an increasing sequence, we have

$$\frac{\lambda \widetilde{F}(y)}{\widetilde{F}(\lambda y)} \leq \lim \inf_{n \to \infty} \frac{a_{\lfloor \lambda n \rfloor}}{a_n} \leq \lim \sup_{n \to \infty} \frac{a_{\lfloor \lambda n \rfloor}}{a_n} \leq 1.$$

Replacing λ by $\frac{1}{\lambda}$, we conclude that for all $\lambda \in (0, \infty)$, $\frac{a_{\lfloor \lambda n \rfloor}}{a_n}$ is uniformly bounded away from 0 and uniformly bounded from above.

Let $\phi(\lambda)$ be defined for each $\lambda \in (0, \infty)$ as a sub-sequential limit of $\frac{a_{\lfloor \lambda n \rfloor}}{a_n}$, that is choose some ($\lambda$-dependent) sub-sequence $t_n \to \infty$, set $\phi(\lambda) = \lim_{n \to \infty} \frac{a_{\lfloor \lambda t_n \rfloor}}{a_{t_n}}$. From the above reasoning we know that $\phi(\lambda) \in (0, \infty)$. Consider the equation

$$\widetilde{F}_{\lfloor \lambda n \rfloor}(y) = \frac{a_{\lfloor \lambda n \rfloor}}{a_n} \frac{\det(B'_{a_{\lfloor \lambda n \rfloor}})/\det(B'_{a_n})}{\lfloor \lambda n \rfloor / n} \widetilde{F}_n\left(\frac{\lfloor \lambda n \rfloor / n}{\det(B'_{a_{\lfloor \lambda n \rfloor}})/\det(B'_{a_n})} y\right),$$

and take the limit along the sub-sequence $\{t_n\}$. We can indeed take the limit on the right hand side of the equation because $\widetilde{F}$ is continuous, and convergence of $\widetilde{F}_n(y) \to \widetilde{F}(y)$ and $B'_n(B'_{\lfloor nt \rfloor})^{-1} \to t^{-E}$ are uniform over compact sets. This yields

$$\widetilde{F}(y) = \phi(\lambda) \frac{\phi(\lambda)^{\operatorname{tr} E}}{\lambda} \widetilde{F}\left(\frac{\lambda}{\phi(\lambda)^{\operatorname{tr} E}} y\right).$$

Note that the function

$$z \to \frac{z^{\operatorname{tr} E}}{\lambda} \widetilde{F}\left(\frac{\lambda}{z^{\operatorname{tr} E}} y\right)$$

is non-decreasing because $y \to \frac{\widetilde{F}(y)}{y}$ is non-increasing. As $z \to \frac{\widetilde{F}(y)}{z}$ is strictly decreasing, the solution $z_0 = z(\lambda, y)$ to

$$\frac{\widetilde{F}(y)}{z} = \frac{z^{\operatorname{tr} E}}{\lambda} \cdot \widetilde{F}\left(\frac{\lambda}{z^{\operatorname{tr} E}} y\right)$$

is unique. Hence limit $\phi(\lambda) = \lim_{n\to\infty} \frac{a_{\lfloor \lambda n \rfloor}}{a_n}$ exists in $(0, \infty)$ for all $\lambda \in (0, \infty)$.

Observe that

$$\begin{aligned}
\phi(\lambda_1\lambda_2) &= \lim_{n\to\infty} \frac{a_{\lfloor \lambda_1\lambda_2 n \rfloor}}{a_n} \\
&= \lim_{n\to\infty} \frac{a_{\lfloor \lambda_1\lambda_2 n \rfloor}}{a_{\lfloor \lambda_2 n \rfloor}} \cdot \frac{a_{\lfloor \lambda_2 n \rfloor}}{a_{\lfloor n \rfloor}} \\
&= \phi(\lambda_1)\phi(\lambda_2).
\end{aligned}$$

Therefore ϕ is multiplicative and

$$\phi(\lambda) = \lambda^{\kappa}$$

with $\kappa = \log_2 \phi(2)$. Plugging this back in

$$\widetilde{F}(y) = \phi(\lambda) \cdot \frac{\phi(\lambda)^{\operatorname{tr} E}}{\lambda} \cdot \widetilde{F}\left(\frac{\lambda}{\phi(\lambda)^{\operatorname{tr} E}} y\right),$$

we have

$$\widetilde{F}(y) = \lambda^{\kappa} \cdot \frac{\lambda^{\kappa \operatorname{tr} E}}{\lambda} \cdot \widetilde{F}\left(\frac{\lambda}{\lambda^{\kappa \operatorname{tr} E}} y\right).$$

Setting $y = 1$ gives

$$\widetilde{F}(1) = \lambda^{\kappa + \kappa \operatorname{tr} E - 1} \cdot \widetilde{F}(\lambda^{1 - \kappa \operatorname{tr} E}),$$

so that

$$\widetilde{F}(y) = \widetilde{F}(1) y^{\frac{1-\kappa \operatorname{tr} E - \kappa}{1 - \kappa \operatorname{tr} E}}.$$

The fact that

$$\lim_{n\to\infty} \frac{\log a_n}{\log n} = \kappa,$$

follows exactly from the reasoning in [2]. □

6.2 Discussion of the constant

In this subsection, we follow the truncation argument in [6] to prove Lemma 5.5.

Let

$$J := \sup_{\lambda > 0} \inf_{f \in \mathcal{F}_{\mathbb{T}}} \left\{ \lambda^{-1} \mathcal{E}_{\widetilde{\eta}} \left(\sqrt{f}, \sqrt{f} \right) + \lambda^{(1-\gamma) \operatorname{tr} E} \int_{\mathbb{T}} \widetilde{F}(f(x)) dx \right\},$$

let $\epsilon > 0$ be an arbitrary small number. We need to find $\Omega \in \mathcal{G}$ and $g \in \mathcal{F}_\Omega$ such that

$$\mathcal{E}_\eta \left(\sqrt{g}, \sqrt{g} \right) + \int_\Omega \widetilde{F}(g(x)) dx \leq J + \epsilon.$$

Choose λ sufficiently large (we will specify how large we need λ to be in the end), by definition of J, there exists $f \in \mathcal{F}_{\mathbb{T}}$ such that

$$\lambda^{-1} \mathcal{E}_{\widetilde{\eta}}(\sqrt{f}, \sqrt{f}) + \lambda^{(1-\gamma) \operatorname{tr} E} \int_{\mathbb{T}} \widetilde{F}(f(x)) dx < J + \frac{\epsilon}{2}.$$

We think of functions on $\mathbb{T}$ also as functions on fundamental domain $[0,1)^d$. Following Lemma 3.4 [6], let

$$E^\lambda = \cup_{i=1}^d \{ (0 \leq x_i \leq \frac{1}{\sqrt[4]{\lambda}}) \cup (1 - \frac{1}{\sqrt[4]{\lambda}} \leq x_i < 1) \},$$

there exists $a \in \mathbb{T}$ such that translated function $f_a(x) = f(x-a)$ satisfies

$$\int_{E^\lambda} f_a dx \leq \frac{2d}{\sqrt[4]{\lambda}}.$$

Because of translation invariance of the expression on the torus, we can replace f by f_a in the expression without changing the value. Therefore we may assume that $f \in \mathcal{F}_{\mathbb{T}}$ satisfies

$$\lambda^{-1} \mathcal{E}_{\widetilde{\eta}}(\sqrt{f}, \sqrt{f}) + \lambda^{(1-\gamma) \operatorname{tr} E} \int_{\mathbb{T}} \widetilde{F}(f(x)) dx < J + \frac{\epsilon}{2},$$

and

$$\int_{E^\lambda} f dx \leq \frac{2d}{\sqrt[4]{\lambda}}.$$

Take a bump function ϕ_0 on $\mathbb{R}$ defined as

$$\phi_0(x) = \begin{cases} 0 & for & x \notin [0,1] \\ \sqrt[4]{\lambda} \cdot x & for & x \in [0, \frac{1}{\sqrt[4]{\lambda}}] \\ 1 & for & x \in [\frac{1}{\sqrt[4]{\lambda}}, 1 - \frac{1}{\sqrt[4]{\lambda}}] \\ \sqrt[4]{\lambda}(1-x) & for & x \in [1 - \frac{1}{\sqrt[4]{\lambda}}, 1] \end{cases}$$

and let $\widehat{\phi_0}(x_1, ..., x_d) = \phi_0(x_1) ... \phi_0(x_d)$, and $\psi(x) = \widehat{\phi_0}(x)^2$. Then $\left\| \triangledown \widehat{\phi_0} \right\| \leq \sqrt{d} \cdot \sqrt[4]{\lambda}$.

Let $T_\lambda := \lambda^E \left([0,1)^d \right)$, that is the image of the fundamental domain $[0,1)^d$ under transformation λ^E. Under the map

$$\lambda^E : [0,1)^d \to T_\lambda,$$

let f_λ denote the density function of the push-forward measure of $f(x)dx$ on $[0,1)^d$. Change of variable formula, scaling property $t^E(W) = t \cdot W$ of the Lévy measure and $\widetilde{F}(y) = \widetilde{F}(1)y^\gamma$ together give that

$$\begin{aligned}
& \lambda^{-1}\mathcal{E}_{\widetilde{\eta}}\left(\sqrt{f}, \sqrt{f}\right) + \lambda^{(1-\gamma)\operatorname{tr} E} \int_{\mathbb{T}} \widetilde{F}(f(x))dx \\
= \quad & \mathcal{E}_{\lambda,\widetilde{\eta}}\left(\sqrt{f}, \sqrt{f}\right) + \int_{T_\lambda} \widetilde{F}(f_\lambda(x))dx,
\end{aligned}$$

where $\mathcal{E}_{\lambda,\widetilde{\eta}}$ is the Dirichlet form of the process projected to $\mathbb{R}^d/(\lambda^E\mathbb{Z}^d)$.We can also push-forward bump functions $\widehat{\phi_0}$ and ψ to functions on T_λ,

$$\begin{aligned}
\widehat{\phi_\lambda}(x) \quad &: \quad = \widehat{\phi_0}(t^{-E}x) \\
\psi_\lambda(x) \quad &: \quad = \psi(t^{-E}x).
\end{aligned}$$

Claim 6.1. *Take* $\Omega = \lambda^E(0,1)^d$,

$$g(x) := \frac{f_\lambda(x)\psi_\lambda(x)}{\int_{\mathbb{R}^d} f_\lambda(x)\psi_\lambda(x)dx}$$

gives us the function we are looking for.

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